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THEORY

OF THE

MOON'S MOTION.

BY

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PREFACE.

An abstract of the following investigation was published in Nos. (2024–2026) of the *Astronomische Nachrichten*. But, a few verbal inaccuracies, together with numerous typographical errors, being found in the work as printed, it was decided to reprint the whole investigation, giving the mathematical developments more in detail, and also developing the formula for the latitude of the moon, a subject not touched upon in that investigation. The author hopes to find leisure for the complete development of the perturbations of the moon's motions, by means of the differential equations given in this first chapter. It is believed that these equations will give the co-ordinates of the moon in a more direct and simple manner than any combination of the general differential equations of motion hitherto employed for that purpose.

THEORY OF THE MOON'S MOTION.

INTRODUCTION.

1. IT is now nearly two centuries since the theory of the universal gravitation of matter according to the law of the inverse square of the distances was discovered and subjected to calculation by NEWTON. The theory when regarded as a law of nature may be stated as follows: *Every particle of matter in the universe attracts every other particle with a force which varies directly as its mass, and inversely as the square of the distance between them.*

Mathematicians have attempted to deduce from this law of matter all the phenomena attending the motions of the heavenly bodies. For this purpose the sun, planets and satellites may be regarded as particles of matter when compared with the distances which separate them from the other bodies of the system, the masses of the planets being regarded as infinitely small in comparison with that of the sun. The principal mathematical consequences resulting from the operation of this law of matter may be stated as in the three following theorems:

I. *The orbits of the planets and comets are conic sections in which the sun occupies the principal focus;*

II. *The radius vector of each planet or comet sweeps over equal areas in equal times;* and

III. *The squares of the times of revolution of the different planets are to each other in the same proportion as the cubes of their mean distances from the sun.*

These three laws, which were discovered by KEPLER, may be regarded as the embodiment of the theory of universal gravitation, and are the foundation of physical or mathematical astronomy. They are, however, exact only on the supposition that the masses of the planets are infinitely small in comparison with that of the sun, and hence are not applicable to the solar system without some modification, because the masses of some of the planets are finite instead of being infinitely small. The motions of the planets therefore do not strictly conform to

the preceding laws, for, in obeying their mutual attraction, they must deviate a little from the elliptical paths which they would exactly follow if they were attracted only by the sun Not only are the planets disturbed in their motions around the sun by reason of their mutual attraction, but the satellites are also disturbed in their motions around their primaries by the attraction of the sun and by the other planets of the system The smallness of the planetary masses in comparison with that of the sun, however, permits these bodies to conform very nearly to the laws of the elliptical motion, and hence the calculation of their perturbations is not a difficult problem But in the motions of the satellites about their primaries, the principal disturbing body is the sun, and the effect of his attraction is too great to be overlooked Especially is this the case in regard to the motion of the moon around the earth, the disturbing force of the sun being so great as to cause the transverse axis of her orbit to describe a complete revolution in about nine years, and also causing the nodes of the orbit to revolve in about double that interval

2 The mathematical theory of the perturbations of the planets and satellites develops the laws according to which the various forces operate in the disturbance of each other's movements They may be classed as follows

First The operation of a comparatively large force during a short interval of time,

Second The operation of a comparatively small force during a long period of time, and

Third A very slow change in the mean motions of the different bodies arising from the variations of the elements of their orbits produced by their mutual attractions

The first class produces the periodic inequalities, which depend solely on the mutual distances and configurations of the different bodies the second class produces the inequalities of long period, which may be said to depend on the configuration of the elements of their motions, and the third class produces the secular inequalities The second and third classes affect only the mean motions

3 The problem of deducing all the circumstances which affect the motion of the different bodies of the system, directly from the principle of universal gravitation, has, perhaps, more than any other exercised the ingenuity of mathematicians and astronomers during the last century and a half The theory of the periodic and secular inequalities of the planets were developed in sufficient detail for the purposes of astronomy about the close of the last century Considerable progress was also made in the development of the lunar theory by LAPLACE, but

the magnitude of the disturbing forces being so much greater than those which disturb the planetary motions, the labor necessary for the construction of a perfect lunar theory is vastly increased. LA PLACE states that he has determined all the inequalities of the first, second and third orders, and the most important ones of the fourth order, and has continued the approximation to quantities of the fourth order inclusively, and also retained those of the fifth order, which arise in the calculation. But much more elaborate theories of the moon's motion were published during the first half of the present century, in which the approximations are carried to terms of a much higher order than was attempted by LA PLACE; the most complete and perfect, being that of PLANA, in which the approximations are carried to terms of the seventh order, was published in the year 1832. PONTÉCOULANT also investigated the lunar theory in the fourth volume of his *Théorie Analytique du Système du Monde*, which was published in the year 1846, in which he has also carried the approximations to terms of the seventh order. The theories of the moon's motion by PLANA and PONTÉCOULANT were produced by entirely different methods of development—the one by employing the true longitude as the independent variable, and the other using the mean longitude for the same purpose—and when reduced to the same form were found to be almost identically the same, the coefficients of the inequalities seldom differing from each other by so much as one second of arc. It would therefore seem that the calculations of both had been correctly made. But perhaps the most elaborate and complete investigations on the theory of the moon's motion are those of HANSEN and DELAUNAY. HANSEN published very elaborate "*Tables de la Lune*" in the year 1857; but the analysis by which he obtained his formulæ has not, so far as I am informed, yet been published. The lunar theory by DELAUNAY was published in two large quarto volumes, which appeared in the years 1860 and 1867, and is perhaps the most complete and perfect work on the lunar theory that has yet been given to astronomers.

4. In a valuable and interesting discussion on the apparent inequalities of long period in the motion of the moon, published in the *American Journal of Science and Arts* for September, 1870, Professor NEWCOMB makes the following statement in regard to the inequalities of short period in the motion of the moon:

"The problem of determining the motion of the moon around the earth under the influence of the combined attraction of the sun and planets has, more than any other, called forth the efforts of mathematicians and astronomers. Nearly every great geometer since NEWTON has added something to the simplicity or the accuracy of the solution, and in our own day we have seen it successfully com-

pleted in its simplest form, in which the earth, the moon and the sun are regarded as material points, moving under the influence of their mutual attractions The satisfactory solutions are due to the genius of HANSEN and of DELAUNAY Working independently of each other, each using a method of his own invention more rigorous than had before been applied, they arrived at expressions for the longitude of the moon which, being compared, were found to exhibit an average discrepancy of less than a second of arc No doubt could remain of the substantial correctness of each'

This statement, coming from so eminent an astronomer, would lead us to suppose that the mathematical theory of the moon's motion, in so far as it depends on the attraction of the sun and planets, may be regarded as perfect Four years later the same writer, after a thorough and extended examination of the question, assures us of the correctness of his conclusions

HANSEN has also given in his tables of the moon two inequalities of long period depending on the action of *Venus* DELAUNAY, in computing the coefficients of these two inequalities by his own method, reproduces, very nearly, HANSEN'S coefficient of one of them, but finds the other to be insensible

Professor NEWCOMB then proceeds to inquire whether we have in either theory a satisfactory agreement with observations, and finally arrives at the conclusion that the problem of the inequalities of long period in the moon's mean motion is really no nearer such a solution as will agree with observation than when it was left by LA PLACE, and he is obliged to resort to the operation of irregular and extraneous causes, the data for the accurate determination of which are wanting, in order to fill up the gap between theory and observation

In the *Monthly Notices of the Royal Astronomical Society* for November 14, 1873, Mr AIRY also enters into a discussion in regard to the inequalities of long period in the moon's mean motion, and states his conclusion in these words "I conceive it to be totally impossible that a complete and correct theory can leave such large and systematic discordances And I express my opinion that there is some serious defect in the Lunar Theory'

These statements, coming from such eminent astronomers, are sufficient to encourage the belief that the published theories of the inequalities of long period in the moon's mean motion possess but very little value

5 Let us now inquire whether the inequalities of short period in the moon's motion are any more perfectly represented by existing theories than the inequalities of long period The theories of HANSEN and DELAUNAY, according to Professor NEWCOMB, may be regarded as identical in their results, and they also

contain all the important researches of astronomers up to a very recent date. The theories of PLANA and PONTÉCOULANT may also be regarded as identical in their results; the former was published in 1832, and "*Tables of the Moon,*" founded on it, were published by Professor PEIRCE in 1853. PLANA'S theory was somewhat modified by the theoretical investigations of HANSEN; and some empirical corrections, proposed by AIRY and LONGSTRETH, were also introduced; so that the tables in their present shape do not exhibit the results of pure theory alone. But if the empirical corrections introduced are legitimate parts of a correct lunar theory, and only await the demonstrations of the physical astronomer, we may assume that they have a permanent value, and that the tables are permanently improved by their introduction; otherwise they possess only a temporary value, and will vitiate the tables after a while to the extent of their primitive improvement. Now, the ephemeris of the moon, contained in the *American Ephemeris and Nautical Almanac*, is derived from PEIRCE'S tables of the moon, and we have only to consult the volumes of the Naval Observatory at Washington in order to learn to what extent the moon deviates from the requirements of theory. In this comparison we may suppose that the moon's mean place is correctly given by the tables, in which case the observed deviations between theory and observation will serve to indicate the magnitude of one or more inequalities of short period in the moon's motion which analysis has yet failed to point out. We shall therefore find that, in the year 1856, the moon's place differed from the computed place between the limits $+10''.4$ and $-12''.4$; in 1857 the errors were between the limits $+7''.7$ and $-10''.4$; in 1858 they were between $+14''.4$ and $-9''.3$; in 1861 they were $+10''.5$ and $-9''.6$; in 1862 they were $+9''.0$ and $-11''.3$; in 1871 they were $+8''.4$ and $-4''.8$; and in 1872 they were $+14''.0$ and $-12''.0$. These residuals of course show only the observed deviation of the moon from her computed place, and do not necessarily indicate the maximum extent to which the theory deviates from the true place of the moon, as cloudy weather no doubt frequently prevents observations at times when the deviations are at their maximum values. The residuals above given would seem to indicate that the mean place of the moon was very closely given by the tables, since the positive and negative errors are very nearly equal to each other. According to Professor NEWCOMB, HANSEN'S tables represented the moon's place at the time of their publication and for a few years after much more closely than PEIRCE'S. In the year 1862 the errors of HANSEN'S tables were included within the limits of $+4''.0$ and $-9''.0$; in 1871 the limits were $+2''.5$ and $-16''.4$; while in 1872 they were $-0''.2$ and $-16''.2$; thus indicating a narrower range of periodic

variations, and also a much wider range in a progressive mean motion It should be observed, however, that the preceding comparison exhibits only errors of right ascension, and it is probable that the errors in longitude would generally be some what larger

If we in like manner compare the observed and computed longitudes of the moon as given in the *Reductions of the Greenwich Lunar Observations from* 1831 *to* 1851, after correcting for the error of mean longitude, we shall find an oscillation between theory and observation amounting to an average value of about 16″

LA PLACE says, in the Introduction to his *Theory of the Moon*, that the error of the tables formed from his theory will very rarely exceed 32″, and he also states that the astronomer BURG, by deriving the forms all the arguments from theory, and rectifying the coefficients by means of numerous observations, had constructed tables of the moon's motion whose greatest errors were less than 13″ If this statement of LA PLACE, which was made three quarters of a century ago, was borne out by the observations of the moon made at that time, it would indicate a degree of perfection in the lunar tables, with respect to the inequalities of short period, which is scarcely exceeded by the tables in use at the present time

6 From the preceding comparison it would appear that the lunar theories of HANSEN and DELAUNAY, when measured by the criterion to which all physical theories must be subjected, are but little, if any, more perfect than the theories of PLANA and PONTÉCOULANT, which preceded them by twenty years, and we also perceive that the lunar theory employed in the *Reductions of the Greenwich Lunar Observations*, and also the tables in use at the close of the last century, in so far as the inequalities of short period are concerned, are but little inferior to the tables in use at the present time It is also possible that the superiority of recent tables over those in use half a century ago may arise more from the corrections to the mean elements of her motion than from any improvement in the theory of her perturbations It would thus appear either that the theory of gravitation had been imperfectly or incorrectly developed, or else that the moon undergoes perturbations from the action of other forces than gravitation

7 The residuals above given would seem to indicate that our present lunar tables, instead of being correct to terms of the seventh order, are really erroneous by some of the smaller terms of the third order, and as the writer had, previous to making this comparison, assured himself, by a careful examination of the mathematical theory of her motion, that some terms of the third order had been overlooked, he does not hesitate to announce the fact to astronomers, and he

confidently believes that a correct theory developed to terms of the fifth order will be found to represent the motions of the moon, in so far as the inequalities of short period are concerned, with far greater precision than any published at the present day.

8. It is true that we have the combined assurance of all the great mathematicians of the present century who have given especial attention to the subject that no inequalities have been overlooked, and that the theory possesses all the accuracy claimed for it. To this we would reply that the error is fundamental and precedes any development of the perturbing function, simply growing out of the latitude of the perigee in the development of the undisturbed elliptical motion. If this statement is correct, it is easy to perceive that the fundamental error would necessarily ramify the whole development of the perturbing function and vitiate more or less all the conclusions deduced from it. Nor do we regard the argument that the agreement of so many profound mathematicians, that the theory of gravitation when legitimately applied to the moon's motion would produce the existing lunar theories, as possessing much weight. Such evidence is merely negative in its character, and possesses no importance whatever when coming in conflict with positive evidence to the contrary. If any number of persons develop any mathematical expression by as many different methods, and all obtain identically the same result, it is satisfactory proof that the development has been correctly made; but it is no more certain or satisfactory than it would be if a single person had made all the developments by as many different methods. Besides, if an equation is supposed to possess a certain physical or geometrical property, the development of the equation into an infinite series, or in any other form whatever, must also possess the same property. But no correct development or transformation of an equation can introduce any new element, either physical or geometrical. The development must reproduce all the errors or infirmities of the function from which it was derived. The assumption that a given mathematical expression possesses a certain physical or geometrical property should therefore be fully justified by a rigorous discussion before such expression is made the basis of a physical theory. It is to this point in the lunar theory that we would now call attention.

9. For this purpose let us take the fundamental equations for the latitude and reciprocal of the projected radius vector of the moon which are used by La Place and Plana as the basis of their respective theories of the moon's motion. If we denote the mean distance and eccentricity of the moon's orbit by a and e, the longitudes of the perigee and node by ω and Ω, the inclination of the

orbit to the fixed plane by $\imath$, and the longitude, latitude and radius vector by v, θ and r, and also put

$$\gamma = \tan \imath, \qquad s = \tan \theta, \tag{1}$$

we shall have the following equations, in which u denotes the reciprocal of the projected radius vector

$$s = \gamma \sin(v - \Omega), \tag{2}$$

$$u = \frac{1}{a(1-e^2)}\{\sqrt{1+s^2} + e\cos(v-\omega)\} = \frac{1}{r\cos\theta} \tag{3}$$

La Place and Plana have supposed that equation (3) is the equation of a projected ellipse, and we shall now proceed to show that it is not such an equation except for the particular case in which the transverse axis lies in the plane of projection

Equation (2) gives

$$\sqrt{1+s^2} = \sqrt{1+\gamma^2\sin^2(v-\Omega)} = \sqrt{1+\tan^2\theta} = \frac{1}{\cos\theta} \tag{4}$$

Whence,
$$\cos\theta = \frac{1}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}}, \tag{5}$$

$$\sin\theta = \frac{\gamma\sin(v-\Omega)}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}} \tag{6}$$

Now, since $u = \dfrac{1}{r\cos\theta}$, we shall evidently have the maximum and minimum values of u when r and $\cos\theta$ are respectively a minimum and maximum Now, the minimum value of r in an ellipse is $r = a(1-e)$, and equation (5) gives $\cos\theta = \dfrac{1}{\sqrt{1+\gamma^2}}$, when $\cos\theta$ is a minimum, and the maximum values of r and $\cos\theta$ are $a(1+e)$ and 1, respectively, we shall therefore find

$$\text{maximum value of } u = \frac{\sqrt{1+\gamma^2}}{a(1-e)}, \tag{7}$$

and
$$\text{minimum value of } u = \frac{1}{a(1+e)} \tag{8}$$

If we now substitute the value of $\sqrt{1+s^2}$ in equation (3), it will become

$$\frac{1}{r\cos\theta} = u = \frac{1}{a(1-e^2)}\{\sqrt{1+\gamma^2\sin^2(v-\Omega)} + e\cos(v-\omega)\}, \tag{9}$$

and this equation gives

$$\text{maximum value of } u = \frac{\sqrt{1+\gamma^2}+e}{a(1-e^2)}, \tag{10}$$

and
$$\text{minimum value of } u = \frac{1}{a(1+e)}. \tag{11}$$

It will thus be seen that while the two minimum values of u agree, the two maximum values differ by the quantity $\frac{1}{a}(\frac{1}{2}e\gamma^2)$ very nearly, or by a quantity of the third order. We shall soon see that the reason of this coincidence of the minimum values arises from the fact that the transverse axis of the orbit is in the plane of projection, in which case equation (9) is correct.

If we now multiply equation (9) by $\cos\theta = \frac{1}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}}$, we shall find

$$\frac{1}{r} = \frac{1}{a(1-e^2)}\left\{1 + \frac{e\cos(v-\omega)}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}}\right\}. \tag{12}$$

In an ellipse the maximum and minimum values of r are $a(1+e)$ and $a(1-e)$, respectively. If then in equation (12) we suppose that $\omega = \Omega$, we shall find the maximum and minimum values of r to be $a(1+e)$ and $a(1-e)$, as they evidently should be; but if we suppose the ellipse to revolve in its own plane through an angle of 90°, ω will become $\Omega + 90°$, and equation (12) will become

$$\frac{1}{r} = \frac{1}{a(1-e^2)}\left\{1 + \frac{e\sin(v-\Omega)}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}}\right\}, \tag{13}$$

and this equation gives

$$\text{maximum value of } r = a(1-e^2) \div \left\{1 - \frac{e}{\sqrt{1+\gamma^2}}\right\}, \tag{14}$$

and
$$\text{minimum value of } r = a(1-e^2 \div \left\{1 + \frac{e}{\sqrt{1+\gamma^2}}\right\}. \tag{15}$$

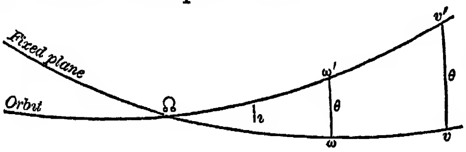
It will be easily seen that the maxima and minima of r derived from equation (13) differ from the true elliptical maxima and minima by quantities of the third order.

Let us now put the first differential coefficient of equation (12) equal to nothing, and we shall find,

$$\frac{dr}{r^2dv} = \frac{\sin(v-\omega)}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}} + \frac{\gamma^2\cos(v-\omega)\cos(v-\Omega)\sin(v-\Omega)}{\{1+\gamma^2\sin^2(v-\Omega)\}^{\frac{3}{2}}} = 0. \tag{16}$$

It is evident that this equation cannot be satisfied when $v-\omega=0$ unless $v-\Omega$ also equal 0° or 90°, which requires that the transverse axis of the orbit should be in the line of the nodes or in a line perpendicular to it; and as this is not the case in nature, it follows that the equation is not applicable to the problem.

10 Let us now transform the equation of the ellipse in such manner that the anomalies, instead of being measured on the plane of the orbit, may be measured on the fixed plane which is inclined to the plane of the orbit by the angle i, and let the place of the ascending node of the orbit on the fixed plane be denoted by Ω, while the quantities v, θ and ω have the same significance as in the last article, also let v' and ω' denote the place of the moon and of the perigee in the orbit, and θ_0 the latitude of the perigee Reference to the figure will make the geometrical conception clear



The equation of the ellipse will evidently give

$$\frac{1}{r} = \frac{1 + e\cos(v' - \omega')}{a(1 - e^2)}, \tag{17}$$

and the required transformation consists in finding the value of $\cos(v' - \omega')$ in terms of the angle $v - \omega$ Now, if we put $v' - \omega' = (v' - \Omega) - (\omega' - \Omega)$, we shall have

$$\cos(v' - \omega') = \cos(v' - \Omega)\cos(\omega' - \Omega) + \sin(v' - \Omega)\sin(\omega' - \Omega) \tag{18}$$

We also have

$$\tan i = \gamma, \qquad \tan\theta = \gamma\sin(v - \Omega), \tag{19}$$

$$\cos\theta = \frac{1}{\sqrt{1 + \gamma^2\sin^2(v - \Omega)}}, \qquad \sin\theta = \frac{\gamma\sin(v - \Omega)}{\sqrt{1 + \gamma^2\sin^2(v - \Omega)}}, \tag{20}$$

$$\cos\theta_0 = \frac{1}{\sqrt{1 + \gamma^2\sin^2(\omega - \Omega)}}, \qquad \sin\theta_0 = \frac{\gamma\sin(\omega - \Omega)}{\sqrt{1 + \gamma^2\sin(\omega - \Omega)}} \tag{21}$$

Then from the right angled spherical triangles of the figure it is easy to deduce

$$\cos(v' - \Omega) = \cos(v - \Omega)\cos\theta, \qquad \sin(v' - \Omega) = \sin\theta\operatorname{cosec} i, \tag{22}$$

$$\cos(\omega' - \Omega) = \cos(\omega - \Omega)\cos\theta_0, \qquad \sin(\omega' - \Omega) = \sin\theta_0\operatorname{cosec} i \tag{23}$$

If we substitute these values in equation (18), it will become

$$\cos(v' - \omega') = \cos(v - \Omega)\cos(\omega - \Omega)\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\operatorname{cosec}^2 i \tag{24}$$

Substituting the values of $\sin\theta_0$ and $\cos\theta_0$ in this equation, it becomes

$$\cos(v' - \omega') = \frac{\cos(v - \Omega)\cos(\omega - \Omega)\cos\theta + \gamma\sin\theta\sin(\omega - \Omega)\operatorname{cosec}^2 i}{\sqrt{1 + \gamma^2\sin^2(\omega - \Omega)}} \tag{25}$$

Now we have $\qquad \gamma\sin\theta = \gamma^2\cos\theta\sin(v - \Omega), \qquad (26)$

and $\qquad \gamma^2\operatorname{cosec}^2 i = \tan^2 i\operatorname{cosec}^2 i = 1 + \tan^2 i = 1 + \gamma^2 \qquad (27)$

If we now substitute the value of $\gamma \sin \theta$ in equation (25), and then the value of $\gamma^2 \operatorname{cosec}^2 i$, we shall obtain, after multiplying by e and putting

$$\cos \theta_0 = \frac{1}{\sqrt{1+\gamma^2 \sin^2 (\omega - \Omega)}},$$

$$\left.\begin{aligned} e \cos (v' - \omega') &= e \cos \theta_0 \cos \theta \cos (v - \Omega) \cos (\omega - \Omega) \\ &\quad + e(1+\gamma^2) \cos \theta_0 \cos \theta \sin (v - \Omega) \sin (\omega - \Omega), \\ &= e\,(1 + \tfrac{1}{2}\gamma^2) \cos \theta_0 \cos \theta \cos (v - \omega) - \tfrac{1}{2} e\gamma^2 \cos \theta_0 \cos \theta \cos (v + \omega - 2\Omega). \end{aligned}\right\} \quad (28)$$

Substituting this value of $e \cos (v' - \omega')$ in equation (17), it becomes

$$\frac{1}{r} = \frac{1}{a(1-e^2)} \{1 + e(1+\tfrac{1}{2}\gamma^2) \cos \theta_0 \cos \theta \cos (v-\omega) - \tfrac{1}{2} e\gamma^2 \cos \theta_0 \cos \theta \cos (v+\omega-2\Omega)\}, \quad (29)$$

which is the polar equation of the ellipse, in which the polar angle is measured on a plane inclined by an angle whose tangent is γ, to the plane of the ellipse itself.

Let us now discuss this equation and see if it meets all the requirements of the problem. If we substitute the values of $\cos \theta$ and $\cos \theta_0$, in the second member it will become

$$\frac{1}{r} = \frac{1}{a(1-e^2)} \left\{ 1 + \frac{e \cos (v - \omega) - \frac{1}{2} e\gamma^2 \cos (v + \omega - 2\Omega)}{\sqrt{1+\gamma^2 \sin^2 (\omega - \Omega)}\sqrt{1+\gamma^2 \sin^2 (v - \Omega)}} \right\}. \quad (30)$$

If we suppose that $v = \omega$, it will give

$$\frac{a(1-e^2)}{r} = 1 + \frac{e + \frac{1}{2} e\gamma^2 - \frac{1}{2} e\gamma^2 \cos^2 (\omega - \Omega)}{1 + \gamma^2 \sin^2 (\omega - e)} = 1 + \frac{e(1+\gamma^2 \sin^2 (\omega - \Omega)}{1+\gamma^2 \sin^2 (\omega - \Omega)} = 1 + e, \quad (31)$$

whatever be the relative values of ω and Ω.

The maximum and minimum values of r are also equal to $a(1+e)$ and $a(1-e)$, which are their correct values. If we now take the differential coefficient of equation (30), we shall find that it may be reduced to the following form:

$$a(1-e^2) \frac{dr}{r^2 dv} = \frac{e(1+\gamma^2) \cos \theta_0 \sin (v - \omega)}{\{1+\gamma^2 \sin^2 (v - \Omega)\}^{\frac{3}{2}}}. \quad (32)$$

This always becomes equal to nothing when $v = \omega$, and when $v = \omega + 180°$, as it manifestly should, since the radius vector is a maximum or minimum at the extremities of the transverse axis.

If we now divide equation (29) by $\cos \theta = \dfrac{1}{\sqrt{1+\gamma^2 \sin^2 (v - \Omega)}}$, it will become

$$\left\{ \begin{array}{l} u = \dfrac{1}{r\cos\theta} = \dfrac{1}{a(1-e^2)} \{\sqrt{1+\gamma^2\sin^2(v-\Omega)} + e(1+\tfrac{1}{2}\gamma^2)\cos\theta_0\cos(v-\Omega) \\ \qquad\qquad\qquad\qquad - \tfrac{1}{2}e\gamma^2\cos\theta_0\cos(v+\omega-2\Omega)\}, \end{array} \right\} \tag{33}$$

This is the correct value of u corresponding to an elliptical orbit, and if we suppose that $\omega = \Omega$ (in which case $\cos\theta_0 = 1$), it will become

$$u = \frac{1}{a(1-e^2)}\{\sqrt{1+\gamma^2\sin^2(v-\Omega)} + e\cos(v-\omega)\}, \tag{34}$$

which is the same as equation (3), after substituting the value of $\sqrt{1+s^2}$ Equation (33) gives

$$\text{maximum value of } u = \frac{\sqrt{1+\gamma^2}}{a(1-e)}, \tag{35}$$

and

$$\text{minimum value of } u = \frac{1}{a(1+e)}, \tag{36}$$

which are identically the same as before found in equations (7) and (8)

We thus see that equations (30) and (33) meet all the requirements of the problem, and are therefore correct For the case in which $\omega = \Omega$, they immediately change to the equations used by LA PLACE and PLANA We also see that for the general problem their equation is erroneous by terms of the third order, and hence their development of the lunar theory must also be erroneous by terms of the third order

Having thus shown that the equation which determines the moon's distance from the earth, according to the theories of LA PLACE and PLANA, is not a conic section, except for a particular case, which does not exist in nature, we shall now deduce from the general differential equations of the motion of a body acted on by the forces of gravitation, the true law of its motion when the positions and magnitudes of the different forces are given

CHAPTER I.

GENERAL DIFFERENTIAL EQUATIONS OF THE MOON'S MOTION, WITH THE THEORY OF HER ELLIPTICAL MOTION, AND THE VARIATION OF THE ARBITRARY CONSTANTS.

1. For this purpose let us take the general differential equations of the motion of a body which is acted upon by any number of gravitating forces, which equations are as follows (*Mécanique Céleste* [499], Bowditch's translation):

$$\frac{ddx}{dt^2}=\left(\frac{dQ}{dx}\right), \qquad \frac{ddy}{dt^2}=\left(\frac{dQ}{dy}\right), \qquad \frac{ddz}{dt^2}=\left(\frac{dQ}{dz}\right). \tag{A}$$

In these equations x, y and z denote the rectangular co-ordinates, and the function Q represents all the forces which act upon the body whose motion is required. The time is denoted by t, whose element dt is supposed to be constant. We may put equations (A) under a more convenient form for computation in the following manner. If we denote the sum of the masses of the moon and earth by μ, and all the other forces which act upon the moon by R, the function Q will be given by the equation

$$Q=\frac{\mu}{r}-R. \tag{1}$$

The co-ordinates, x, y and z, of the moon will also be given by the equations

$$x=r\cos\theta\cos v, \qquad y=r\cos\theta\sin v, \qquad z=r\sin\theta. \tag{2}$$

If we substitute these values of x, y, z and Q in equations (A), they will become

$$\left.\begin{aligned}
&\frac{ddr-rdv^2\cos^2\theta-rd\theta^2}{dt^2}+\frac{\mu}{r^2}=-\left(\frac{dR}{dr}\right), && (3)\\
&\frac{2rdrdv\cos^2\theta-2r^2\sin\theta\cos\theta\,dvd\theta+r^2\cos^2\theta\,ddv}{dt^2}=-\left(\frac{dR}{dv}\right), && (4)\\
&\frac{2rdrd\theta+r^2\sin\theta\cos\theta\,dv^2+r^2dd\theta}{dt^2}=-\left(\frac{dR}{d\theta}\right). && (5)
\end{aligned}\right\} \quad (A')$$

The integrals of these equations will be the polar co-ordinates r, v and θ of the body whose motion is required.

If we multiply equation (4) by dt, and integrate, we shall find

$$r^2\cos^2\theta\frac{dv}{dt}=c-\int\left(\frac{dR}{dv}\right)dt, \tag{6}$$

c being an arbitrary constant quantity.

If we now multiply equation (4) by $\tan\theta \sin v dt$, and equation (5) by $\cos v dt$, and take the sum of their products, we shall get, after putting $2\sin^2\theta = 1 - \cos^2\theta + \sin^2\theta$,

$$\left.\begin{array}{l} \dfrac{1}{dt}\left\{\begin{array}{l} 2rdrdv \sin\theta\cos\theta\sin v + r^2dv^2\sin\theta\cos\theta\cos v + r^2ddv\sin\theta\cos\theta\sin v \\ + r^2dvd\theta\cos^2\theta\sin v + 2rdrd\theta\cos v - r^2dvd\theta\sin v + r^2dd\theta\cos v \\ - r^2dvd\theta\sin^2\theta\sin v \end{array}\right\} \\ \qquad = -dt\left\{\left(\dfrac{dR}{dv}\right)\tan\theta\sin v + \left(\dfrac{dR}{d\theta}\right)\cos v\right\} \end{array}\right\} \quad (7)$$

Equation (7) gives by integration,

$$\left.\begin{array}{l} \dfrac{1}{dt}\{r^2dv\sin\theta\cos\theta\sin v + r^2\,d\theta\cos v\} \\ \quad = c' - \displaystyle\int dt\left\{\left(\frac{dR}{dv}\right)\tan\theta\sin v + \left(\frac{dR}{d\theta}\right)\cos v\right\} \end{array}\right\} \quad (8)$$

In like manner, if we multiply equation (4) by $-\tan\theta\cos v dt$, and (5) by $\sin v dt$, we shall get, by integrating the sum of their products,

$$\left.\begin{array}{l} \dfrac{1}{dt}\{r^2d\theta\sin v - r^2dv\sin\theta\cos\theta\cos v\} \\ \quad = c'' + \displaystyle\int dt\left\{\left(\frac{dR}{dv}\right)\tan\theta\cos v - \left(\frac{dR}{d\theta}\right)\sin v\right\}, \end{array}\right\} \quad (9)$$

c' and c'' being arbitrary constant quantities

If we now multiply equations (A′) by $r^2\{dv\cos\theta\sin v + d\theta\sin\theta\cos v\}$, $dr\dfrac{\sin v}{\cos\theta} + 2rdv\cos\theta\cos v$, and $dr\sin\theta\cos v + 2d\theta\cos\theta\cos v$, respectively, and take the sum of the products, we shall obtain

$$\left.\begin{array}{l} \dfrac{1}{dt^2}\left\{\begin{array}{l} 3r^2drd\theta^2\cos\theta\cos v + 2r^3d\theta dd\theta\cos\theta\cos v - r^3d\theta^3\sin\theta\cos v \\ -r^3d\theta^2dv\cos\theta\sin v + 3r^2drdv^2\cos^3\theta\cos v + 2r^3dvddv\cos^3\theta\cos v \\ -3r^3dv^2d\theta\sin\theta\cos^2\theta\cos v - r^3dv^3\cos^3\theta\sin v + 2rdr^2dv\cos\theta\sin v \\ + r^2ddrdv\cos\theta\sin v + r^2drddv\cos\theta\sin v - r^2drdvd\theta\sin\theta\sin v \\ + r^2drdv^2\cos\theta\cos v + 2rdr^2d\theta\sin\theta\cos v + r^2ddrd\theta\sin\theta\cos v \\ + r^2drdd\theta\sin\theta\cos v + r^2drd\theta^2\cos\theta\cos v - r^2drdvd\theta\sin\theta\sin v \end{array}\right\} \\ \qquad + \mu\{\cos\theta\sin v dv + \sin\theta\cos v d\theta\} \\ = -r^2\{dv\cos\theta\sin v + d\theta\sin\theta\cos v\}\left(\dfrac{dR}{dr}\right) - \left\{dr\dfrac{\sin v}{\cos\theta} + 2rdv\cos\theta\cos v\right\}\left(\dfrac{dR}{dv}\right) \\ \qquad\qquad - \{dr\sin\theta\cos v + 2rd\theta\cos\theta\cos v\}\left(\dfrac{dR}{d\theta}\right) \end{array}\right\} \quad (10)$$

Multiplying equations (A′) respectively by $r^2\{d\theta\sin\theta\sin v - dv\cos\theta\cos v\}$,

$-dr\frac{\cos v}{\cos\theta}+2rdv\cos\theta\sin v$, and $dr\sin\theta\sin v+2rd\theta\cos\theta\sin v$, the sum of their products will give

$$\left.\begin{array}{l}\frac{1}{dt^2}\left\{\begin{array}{l}3r^2drd\theta^2\cos\theta\sin v+2r^3d\theta dd\theta\cos\theta\sin v-r^3d\theta^3\sin\theta\sin v\\ +r^3dvd\theta^2\cos\theta\cos v+3r^2drdv^2\cos^3\theta\sin v+2r^3dvddv\cos^3\theta\sin v\\ -3r^3dv^2d\theta\sin\theta\cos^2\theta\sin v+r^3dv^3\cos^3\theta\cos v-2rdr^2dv\cos\theta\cos v\\ -r^2ddrdv\cos\theta\cos v+rdrdvd\theta\sin\theta\cos v+r^2drdv^2\cos\theta\sin v\\ -r^2drddv\cos\theta\cos v+2rdr^2d\theta\sin\theta\sin v+r^2ddrd\theta\sin\theta\sin v\\ +r^2drdd\theta\sin\theta\sin v+r^2drd\theta^2\cos\theta\sin v+r^2drdvd\theta\sin\theta\cos v\end{array}\right\}\\ \quad+\mu\{d\theta\sin\theta\sin v-dv\cos\theta\cos v\}\\ =r^2\{dv\cos\theta\cos v-d\theta\sin\theta\sin v\}\left(\frac{dR}{dr}\right)+\{dr\frac{\cos v}{\cos\theta}-2rdv\cos\theta\sin v\}\left(\frac{dR}{dv}\right)\\ \qquad-\{dr\sin\theta\sin v+2rd\theta\cos\theta\sin v\}\left(\frac{dR}{d\theta}\right)\end{array}\right\}. \quad (11)$$

If we now multiply equations (A′) respectively by $-r^2d\theta\cos\theta$, $2rdv\sin\theta$, and $-dr\cos\theta+2rd\theta\sin\theta$, the sum of their products will give

$$\left.\begin{array}{l}\frac{1}{dt^2}\left\{\begin{array}{l}-2rdr^2d\theta\cos\theta-r^2ddrd\theta\cos\theta-r^2drdd\theta\cos\theta+r^2drd\theta^2\sin\theta\\ +3r^2drd\theta^2\sin\theta+r^3dv^2d\theta\cos^3\theta+2r^3d\theta dd\theta\sin\theta+r^3d\theta^3\cos\theta\\ +3r^2drdv^2\sin\theta\cos^2\theta+2r^3dvddv\sin\theta\cos^2\theta-2r^3dv^2d\theta\sin^2\theta\cos\theta\end{array}\right\}\\ \quad-\mu\cos\theta\, d\theta\\ =r^2d\theta\cos\theta\left(\frac{dR}{dr}\right)-2rdv\sin\theta\left(\frac{dR}{dv}\right)+\{dr\cos\theta-2rd\theta\sin\theta\}\left(\frac{dR}{d\theta}\right)\end{array}\right\}. \quad (12)$$

Lastly, if we multiply equations (A′) respectively by $2dr$, $2dv$ and $2d\theta$, the sum of their products will give

$$\left.\begin{array}{l}\frac{1}{dt^2}\left\{\begin{array}{l}2drddr+2rdrdv^2\cos^2\theta+2rdrd\theta^2+2r^2d\theta dd\theta\\ +2r^2dvddv\cos^2\theta-2r^2dv^2d\theta\sin\theta\cos\theta\end{array}\right\}+2\mu\frac{dr}{r^2}\\ \qquad=-2\left\{\left(\frac{dR}{dr}\right)dr+\left(\frac{dR}{dv}\right)dv+\left(\frac{dR}{d\theta}\right)d\theta\right\}\end{array}\right\}. \quad (13)$$

If we now take the integrals of equations (10), (11), (12) and (13), we shall obtain

$$\left.\begin{array}{l}\left.\begin{array}{l}\frac{1}{dt^2}\{r^3d\theta^2\cos\theta\cos v+r^3dv^2\cos^3\theta\cos v+r^2drdv\cos\theta\sin v+r^2drd\theta\sin\theta\cos v\}\\ \quad-\mu\cos\theta\cos v-f\end{array}\right\}\\ =-\int\left\{r^2\{dv\cos\theta\sin v+d\theta\sin\theta\cos v\}\left(\frac{dR}{dr}\right)+\{dr\frac{\sin v}{\cos\theta}+2rdv\cos\theta\cos v\}\left(\frac{dR}{dv}\right)\right.\\ \qquad\left.+\{dr\sin\theta\cos v+2rd\theta\cos\theta\cos v\}\left(\frac{dR}{d\theta}\right)\right\}\end{array}\right\}; \quad (14)$$

$$\left.\begin{aligned}&\frac{1}{dt^2}\{r^3d\theta^2\cos\theta\sin v+r^3dv^2\cos^3\theta\sin v-r^2drdv\cos\theta\cos v+r^2drd\theta\sin\theta\sin v\}\\&\quad-\mu\cos\theta\sin v-f'\\&=-\int\left\{r^2\{d\theta\sin\theta\sin v-dv\cos\theta\cos v\}\left(\frac{dR}{dr}\right)+\{2rdv\cos\theta\sin v-\frac{\cos v}{\cos\theta}dr\}\left(\frac{dR}{dv}\right)\right.\\&\qquad\left.+\{dr\sin\theta\sin v+2rd\theta\cos\theta\sin v\}\left(\frac{dR}{d\theta}\right)\right\}\end{aligned}\right\}\tag{15}$$

$$\left.\begin{aligned}&\frac{1}{dt^2}\{r^3d\theta^2\sin\theta+r^3dv^2\sin\theta\cos^2\theta-r^2drd\theta\cos\theta\}-\mu\sin\theta-f''\\&=\int\left\{r^2\,d\theta\cos\theta\left(\frac{dR}{dr}\right)-2rdv\sin\theta\left(\frac{dR}{dv}\right)-\{2rd\theta\sin\theta-dr\cos\theta\}\left(\frac{dR}{d\theta}\right)\right\}\end{aligned}\right\},\tag{16}$$

$$\left.\begin{aligned}&\frac{1}{dt^2}\{dr^2+r^2d\theta^2+r^2dv^2\cos^2\theta\}-\frac{2\mu}{r}+\frac{\mu}{a}\\&\qquad=-2\int\left\{\left(\frac{dR}{dr}\right)dr+\left(\frac{dR}{dv}\right)dv+\left(\frac{dR}{d\theta}\right)d\theta\right\}\end{aligned}\right\},\tag{17}$$

f, f', f'' and $\frac{\mu}{a}$ being arbitrary constant quantities to complete the integrals

Equations (6), (8), (9), (14), (15), (16) and (17) are the polar equivalents of equations (P), *Mecanique Céleste* [572]

If we now multiply equation (8) by $\cos v$, and equation (9) by $\sin v$, the sum of their products will give

$$\left.\begin{aligned}\frac{d\theta}{dt}&=\frac{c'\cos v+c''\sin v}{r^2}\\&\quad+\frac{\sin v}{r^2}\int dt\left\{\left(\frac{dR}{dv}\right)\tan\theta\cos v-\left(\frac{dR}{d\theta}\right)\sin v\right\}\\&\quad-\frac{\cos v}{r^2}\int dt\left\{\left(\frac{dR}{dv}\right)\tan\theta\sin v+\left(\frac{dR}{d\theta}\right)\cos v\right\}\end{aligned}\right\}\tag{18}$$

Equation (6) gives

$$\frac{dv}{dt}=\frac{c}{r^2\cos^2\theta}-\frac{1}{r^2\cos^2\theta}\int\left(\frac{dR}{dv}\right)dt\tag{19}$$

If we now multiply equations (14) and (15) by $\sin v$ and $-\cos v$ respectively, and divide the sum of their products by $r^2\cos\theta\frac{dv}{dt}$, we shall get

$$\left.\begin{aligned}
\frac{dr}{dt} &= \frac{f\sin v - f'\cos v}{r^2\cos\theta}\cdot\frac{dt}{dv} \\
&+\frac{\cos v dt}{r^2\cos\theta\, dv}\int\left\{\begin{aligned}
&r^2\{d\theta\sin\theta\sin v - dv\cos\theta\cos v\}\left(\frac{dR}{dr}\right)\\
&+\left\{2rdv\cos\theta\sin v - dr\frac{\cos v}{\cos\theta}\right\}\left(\frac{dR}{dv}\right)\\
&+\{2rd\theta\cos\theta\sin v + dr\sin\theta\sin v\}\left(\frac{dR}{d\theta}\right)
\end{aligned}\right\}\\
&-\frac{\sin v dt}{r^2\cos\theta\, dv}\int\left\{\begin{aligned}
&r^2\{d\theta\sin\theta\cos v + dv\cos\theta\sin v\}\left(\frac{dR}{dr}\right)\\
&+\left\{2rdv\cos\theta\cos v + dr\frac{\sin v}{\cos\theta}\right\}\left(\frac{dR}{dv}\right)\\
&+\{2rd\theta\cos\theta\cos v + dr\sin\theta\cos v\}\left(\frac{dR}{d\theta}\right)
\end{aligned}\right\}
\end{aligned}\right\}. \tag{20}$$

The integrals of equations (18), (19) and (20) will give the latitude, longitude and radius vector of the moon at any time t. To determine the integrals of these equations it is however necessary to know the values of the constant quantities c, c', c'', f, f', f'' and a which were introduced by the integrations, when the function R is equal to nothing, and we shall now proceed to determine them.

2. For this purpose we shall suppose that $R=0$, which gives $\left(\frac{dR}{dr}\right)=0$, $\left(\frac{dR}{dv}\right)=0$, and $\left(\frac{dR}{d\theta}\right)=0$; and equations (6), (8), (9), (14), (15), (16) and (17) will become

$$r^2\cos^2\theta\frac{dv}{dt} = c; \tag{21}$$

$$r^2\sin\theta\cos\theta\sin v\frac{dv}{dt} + r^2\cos v\frac{d\theta}{dt} = c'; \tag{22}$$

$$r^2\sin v\frac{d\theta}{dt} - r^2\sin\theta\cos\theta\cos v\frac{dv}{dt} = c''; \tag{23}$$

$$\frac{1}{dt^2}\left\{\begin{aligned}&r^3d\theta^2\cos\theta\cos v + r^3dv^2\cos^3\theta\cos v\\ &+ r^2drdv\cos\theta\sin v + r^2drd\theta\sin\theta\cos v\end{aligned}\right\} - \mu\cos\theta\cos v = f; \tag{24}$$

$$\frac{1}{dt^2}\left\{\begin{aligned}&r^3d\theta^2\cos\theta\sin v + r^3dv^2\cos^3\theta\sin v\\ &- r^2drdv\cos\theta\cos v + r^2drd\theta\sin\theta\sin v\end{aligned}\right\} - \mu\cos\theta\sin v = f'; \tag{25}$$

$$\frac{1}{dt^2}\{r^3d\theta^2\sin\theta + r^3dv^2\sin\theta\cos^2\theta - r^2drd\theta\cos\theta\} - \mu\sin\theta = f''; \tag{26}$$

$$\frac{1}{dt^2}\{dr^2+r^2d\theta^2+r^2dv^2\cos^2\theta\}-\frac{2\mu}{r}+\frac{\mu}{a}=0 \tag{27}$$

Equations (18), (19) and (20) also become

$$\frac{d\theta}{dt}=\frac{c'\cos v+c''\sin v}{r^2}, \tag{28}$$

$$\frac{dv}{dt}=\frac{c}{r^2\cos^2\theta}, \tag{29}$$

$$\frac{dr}{dt}=\frac{f\sin v-f'\cos v}{r^2\cos\theta}\quad\frac{dt}{dv} \tag{30}$$

If we substitute the value of dt given by (29) in equation (28) it will become

$$\frac{d\theta}{\cos^2\theta}=\frac{c'}{c}\cos v dv+\frac{c''}{c}\sin v dv \tag{31}$$

Equation (31) gives by integration

$$\tan\theta=\frac{c'}{c}\sin v-\frac{c''}{c}\cos v \tag{32}$$

When the moon is at the node $v=\Omega$, and $\theta=0$, therefore equation (32) will give

$$0=\frac{c'}{c}\sin\Omega-\frac{c''}{c}\cos\Omega, \tag{33}$$

therefore

$$\tan\Omega=\frac{c''}{c'} \tag{34}$$

When θ is a maximum it is equal to the inclination of the orbit, and $\frac{d\theta}{dt}=0$, equations (22) and (23) will therefore give, by substituting the value $\frac{dv}{dt}$,

$$c'=c\tan\theta\sin v,\qquad c''=-c\tan\theta\cos v \tag{35}$$

If we square these equations the sum of their squares will give

$$\tan^2\theta=\frac{c'^2+c''^2}{c^2}=\gamma^2 \tag{36}$$

The quantities c, c' and c'' therefore determine the place of the node and the inclination of the orbit, and if we now put

$$\frac{c'}{c}=\gamma\cos\Omega,\qquad \frac{c''}{c}=\gamma\sin\Omega, \tag{37}$$

equation (32) will become

$$\tan\theta = \gamma\{\sin v\cos\Omega - \cos v\sin\Omega\} = \gamma\sin(v-\Omega), \tag{38}$$

which is the same as equation (2), Int , and gives

$$\sin\theta = \frac{\gamma\sin(v-\Omega)}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}}, \qquad \cos\theta = \frac{1}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}} \tag{39}$$

If we now square the values of c, c' and c'', and put their sum equal to h^2, we shall find

$$h^2 = c^2 + c'^2 + c''^2 = \frac{r^2}{dt^2}\{r^2\cos^2\theta dv^2 + r^2 d\theta^2\} \tag{40}$$

This equation gives

$$\frac{r^2\cos^2\theta dv^2 + r^2 d\theta^2}{dt^2} = \frac{h^2}{r^2}. \tag{41}$$

If we substitute this in equation (27) it will become

$$\frac{dr^2}{dt^2} + \frac{h^2}{r^2} - \frac{2\mu}{r} + \frac{\mu}{a} = 0, \tag{42}$$

At the extremities of the transverse axis, $dr = 0$, and equation (42) will give

$$\mu r^2 - 2\mu a r = -ah \tag{43}$$

This equation gives

$$r = a + a\sqrt{1-\frac{h^2}{\mu a}}, \tag{44}$$

and

$$r = a - a\sqrt{1-\frac{h^2}{\mu a}} \tag{45}$$

The sum of these values of r is equal to the transverse axis of the orbit, and their difference is the double of the eccentricity Therefore $2a$ denotes the transverse axis of the orbit, and $\sqrt{1-\frac{h^2}{\mu a}}$ denotes the ratio of the eccentricity to the semitransverse axis or mean distance If we put

$$e = \sqrt{1-\frac{h^2}{\mu a}}, \tag{46}$$

equations (44) and (45) will give

$$r = a(1 \pm e), \tag{47}$$

which are the values of r at the extremities of the transverse axis

If we now denote the longitude of the moon, when r is a minimum, by ω, and also put $dr=0$, and $v=\omega$, in equations (24) and (25), they will become

$$\left[\frac{1}{dt^2}\{r^3 d\theta^2 \cos\theta + r^3 dv^2 \cos^3\theta\} - \mu\cos\theta\right]\cos\omega = f, \tag{48}$$

$$\left[\frac{1}{dt^2}\{r^3 d\theta^2 \cos\theta + r^3 dv^2 \cos^3\theta\} - \mu\cos\theta\right]\sin\omega = f' \tag{49}$$

These two equations give

$$\tan\omega = \frac{f'}{f} \tag{50}$$

If we now take the sum of the squares of the values of f, f' and f'', given by equations (24–26), we shall obtain

$$\left.\begin{array}{l}\frac{1}{dt^4}\{r^6(d\theta^2+dv^2\cos^2\theta)^2 + r^4 dr^2(d\theta^2+dv^2\cos^2\theta)\} \\ \qquad -\frac{2\mu}{dt^2}r^3\{d\theta^2+dv^2\cos^2\theta\}\end{array}\right\} = f^2+f'^2+f''^2-\mu^2 \tag{51}$$

But equation (41) gives

$$d\theta^2 + dv^2\cos^2\theta = h^2\frac{dt^2}{r^4} \tag{52}$$

Substituting this in equation (51) it becomes

$$\frac{h^4}{r^2} + h^2\frac{dr^2}{dt^2} - \frac{2\mu h^2}{r} = f^2+f'^2+f''^2-\mu^2 \tag{53}$$

If in this equation we put $dr=0$, $r=a(1\pm e)$, and $h^2=a\mu(1-e^2)$, it will become

$$f^2+f'^2+f''^2 = \mu^2 e^2 \tag{54}$$

If we now substitute $dt=\dfrac{r^2\cos^2\theta\, dv}{c}$, $r=a(1-e)$, $v=\omega$, $\theta=\theta_0$ and $dr=0$, in equations (24), (25) and (26), they will become

$$\left[\frac{c^2}{a(1-e)}\left\{\frac{d\theta^2}{\cos^4\theta_0 dv^2} + \frac{1}{\cos^2\theta_0}\right\} - \mu\right]\cos\theta_0\cos\omega = f, \tag{55}$$

$$\left[\frac{c^2}{a(1-e)}\left\{\frac{d\theta^2}{\cos^4\theta_0 dv^2} + \frac{1}{\cos^2\theta_0}\right\} - \mu\right]\cos\theta_0\sin\omega = f', \tag{56}$$

$$\left[\frac{c^2}{a(1-e)}\left\{\frac{d\theta^2}{\cos^4\theta_0 dv^2} + \frac{1}{\cos^2\theta_0}\right\} - \mu\right]\sin\theta_0 = f'' \tag{57}$$

But equation (38) gives, when $v=\omega$,

$$d\theta = \gamma\cos^2\theta_0\cos(\omega-\Omega)\,dv, \tag{58}$$

whence we get

$$\frac{d\theta^2}{dv^2\cos^4\theta_0} = \gamma^2\cos^2(\omega-\Omega), \tag{59}$$

and

$$\frac{1}{\cos^2\theta_0} = 1+\gamma^2\sin^2(\omega-\Omega), \tag{60}$$

whence we get

$$\frac{d\theta^2}{\cos^4\theta_0 dv^2} + \frac{1}{\cos^2\theta_0} = 1+\gamma^2, \tag{61}$$

and equations (55–57) become

$$\left\{\frac{c^2(1+\gamma^2)}{a(1-e)} - \mu\right\}\cos\theta_0\cos\omega = f, \tag{62}$$

$$\left\{\frac{c^2(1+\gamma^2)}{a(1-e)} - \mu\right\}\cos\theta_0\sin\omega = f', \tag{63}$$

$$\left\{\frac{c^2(1+\gamma^2)}{a(1-e)} - \mu\right\}\sin\theta_0 = f''. \tag{64}$$

But equations (36), (40) and (46) give

$$c^2(1+\gamma^2) = c^2 + c'^2 + c''^2 = h^2 = a\mu(1-e^2), \tag{65}$$

therefore we get

$$\frac{c^2(1+\gamma^2)}{a(1-e)} = \mu(1+e). \tag{66}$$

Substituting this value in equations (62–64), we get finally,

$$\left.\begin{aligned} f &= \mu e\cos\theta_0\cos\omega, \\ f' &= \mu e\cos\theta_0\sin\omega, \\ f'' &= \mu e\sin\theta_0. \end{aligned}\right\} \tag{67}$$

Therefore the quantities f, f' and f'' denote the product of the sum of the masses of the moon and earth into the co-ordinates of the centre of the orbit when referred to the focus as the origin.

3. Having thus found the values of the constant quantities introduced by the integrations, if we now substitute them in the differential equations of the co-ordinates r, v and θ, we shall obtain, by means of another integration, the values of these co-ordinates in terms of the time. But as we have already found the value of θ in terms of v, in equation (38), we shall also find the values of r and t

in terms of v We shall then, by inverting the formulas, be able to find the values of r, v and θ in terms of the time t

Equation (65) gives

$$c = \frac{\sqrt{\mu a(1-e^2)}}{\sqrt{1+\gamma^2}} \tag{68}$$

If we substitute this value in equation (29) it will become

$$\frac{dt}{dv} = \frac{\sqrt{1+\gamma^2}}{\sqrt{\mu a(1-e^2)}} \quad r^2 \cos^2\theta \tag{69}$$

And if we also substitute these values, and also the values of f and f', in equation (30), it will give

$$\frac{dr}{r^2} = \frac{e(1+\gamma^2)\cos_0}{a(1-e^2)} \quad \cos^3\theta \sin(v-\omega)\,dv \tag{70}$$

But we have
$$\cos^3\theta = \frac{1}{\{1+\gamma^2\sin^2(v-\Omega)\}^{\frac{3}{2}}}, \tag{71}$$

which, being substituted in equation (70), gives

$$\frac{dr}{r^2} = \frac{e(1+\gamma^2)\cos\theta_0 \sin(v-\omega)\,dv}{a(1-e^2)\{1+\gamma^2\sin^2(v-\Omega)\}^{\frac{3}{2}}} \tag{72}$$

This is identically the same as equation (39), Int , and hence equation (72) will give, by integration,

$$\frac{1}{r} = \frac{1}{a(1-e^2)}\left\{1 + \frac{e(1+\frac{1}{2}\gamma^2)\cos\theta_0\cos(v-\omega) - \frac{1}{2}e\gamma^2\cos\theta_0\cos(v+\omega-2\Omega)}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}}\right\}, \tag{73}$$

which is the same as equation (36), Int , obtained from purely geometrical considerations, and is therefore the equation of an ellipse

If we now substitute this value of r in equation (69) it will become

$$\frac{dt}{dv} = \frac{a^{\frac{3}{2}}(1-e^2)^{\frac{3}{2}}\sqrt{1+\gamma^2}\cos^2\theta}{\sqrt{\mu}\left\{1 + \frac{e(1+\frac{1}{2}\gamma)\cos\theta_0\cos(v-\omega) - \frac{1}{2}e\gamma^2\cos\theta_0\cos(v+\omega-2\Omega)}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}}\right\}^2} \tag{74}$$

This equation expresses the rigorous relation which exists between the element

of time dt and the differential of the moon's motion in longitude corresponding to any part of the orbit, and its integral will give the true time t, which is required for the moon to pass through any arc of longitude which is denoted by v It is not, however, readily integrated in its present form, but we may develop the second member into an infinite series arranged according to the ascending powers of e and γ, which, in the lunar theory, are numerically small quantities, and carry the approximations to any degree of accuracy which may be necessary The different terms of the equation when thus developed can then be integrated separately, without any analytical difficulty We shall now attend to this trans formation of equation (74), and shall carry the approximation to terms of the seventh order of magnitude depending on the eccentricity and inclination of the orbit This degree of approximation is greater than is necessary in the theory of the moon's motion, but as the same formula may be applied to the motions of the planets, it was thought best to give it all needful extension for that purpose

4 If we develop the variable part of the second member of equation (74) by the binomial theorem, it will become as follows

$$\cos^2\theta\left\{1+\frac{e(1+\frac{1}{2}\gamma^2)\cos\theta_0\cos(v-\omega)-\frac{1}{2}e\gamma\ \cos\theta_0\cos(v+\omega-2\Omega)}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}}\right\}^{-2}$$

$$\left.\begin{aligned}
&=\cos^2\theta-2\ \frac{e(1+\frac{1}{2}\gamma^2)\cos\theta_0\cos(v-\omega)-\frac{1}{2}e\gamma^2\cos\theta_0\cos(v+\omega-2\Omega)}{\{1+\gamma^2\sin^2(v-\Omega)\}^{\frac{3}{2}}}\\
&\qquad+3\ \frac{\{e(1+\frac{1}{2}\gamma^2)\cos\theta_0\cos(v-\omega)-\frac{1}{2}e\gamma^2\cos\theta_0\cos(v+\omega-2\Omega)\}^2}{\{1+\gamma^2\sin^2(v-\Omega)\}^2}\\
&\qquad-4\frac{\{e(1+\frac{1}{2}\gamma^2)\cos\theta_0\cos(v-\omega)-\frac{1}{2}e\gamma^2\cos\theta_0\cos(v+\omega-2\Omega)\}^3}{\{1+\gamma^2\sin\ (v-\Omega)\}^{\frac{5}{2}}}\\
&\qquad+5\frac{\{e(1+\frac{1}{2}\gamma^2)\cos\theta_0\cos(v-\omega)-\frac{1}{2}e\gamma^2\cos\theta_0\cos(v+\omega-2\Omega)\}^4}{\{1+\gamma^2\sin\ (v-\Omega)\}^3}\\
&\qquad-6\frac{\{e(1+\frac{1}{2}\gamma^2)\cos\theta_0\cos(v-\omega)-\frac{1}{2}e\gamma^2\cos\theta_0\cos(v+\omega-2\Omega)\}^5}{\{1+\gamma^2\sin^2(v-\Omega)\}^{\frac{7}{2}}}\\
&\qquad+7\frac{\{e(1+\frac{1}{2}\gamma^2)\cos\theta_0\cos(v-\omega)-\frac{1}{2}e\gamma^2\cos_0\theta\cos(v+\omega-2\Omega)\}^6}{\{1+\gamma^2\sin^2(v-\Omega)\}^4}\\
&\qquad-8\frac{\{e(1+\frac{1}{2}\gamma^2)\cos\theta_0\cos(v-\omega)-\frac{1}{2}e\gamma^2\cos_0\theta\cos(v+\omega-2\Omega)\}^7}{\{1+\gamma^2\sin^2(v-\Omega)\}^{\frac{9}{2}}}
\end{aligned}\right\}\quad(75)$$

We shall now develop each term of this equation separately Since $\cos\theta = \dfrac{1}{\sqrt{1+\gamma^2\sin^2(v-\Omega)}}$, we shall obtain

$$\cos^2\theta = \{1+\gamma^2\sin^2(v-\Omega)\}^{-1} = 1-\gamma^2\sin^2(v-\Omega)+\gamma^4\sin^4(v-\Omega)-\gamma^6\sin^6(v-\Omega)$$

$$\left.\begin{aligned} &= 1-\tfrac{1}{2}\gamma^2(1-\tfrac{3}{4}\gamma^2+\tfrac{5}{8}\gamma^4)+\tfrac{1}{2}\gamma^2(1-\gamma^2+\tfrac{15}{16}\gamma^4)\cos 2(v-\Omega) \\ &\quad +\tfrac{1}{8}\gamma^4(1-\tfrac{3}{2}\gamma^2)\cos 4(v-\Omega)+\tfrac{1}{32}\gamma^6\cos 6(v-\Omega) \end{aligned}\right\} \tag{76}$$

We also have

$$\cos\theta_0 = \{1+\gamma^2\sin^2(\omega-\Omega)\}^{-\frac{1}{2}}$$

$$\left.\begin{aligned} &= 1-\tfrac{1}{4}\gamma^2+\tfrac{9}{64}\gamma^4-\tfrac{25}{256}\gamma^6+\tfrac{1}{4}\gamma^2(1-\tfrac{3}{4}\gamma^2+\tfrac{75}{128}\gamma^4)\cos 2(\omega-\Omega) \\ &\quad +\tfrac{3}{64}\gamma^4(1-\tfrac{5}{4}\gamma^2)\cos 4(\omega-\Omega)+\tfrac{5}{512}\gamma^6\cos 6(\omega-\Omega) \end{aligned}\right\} \tag{77}$$

We shall therefore obtain the following values of the different terms of the second member of equation (75)

$$\frac{e(1+\frac{1}{2}\gamma^2)\cos\theta_0\cos(v-\omega)-\frac{1}{2}e\gamma^2\cos\theta_0\cos(v+\omega-2\Omega)}{\{1+\gamma^2\sin^2(v-\Omega)\}^{\frac{3}{2}}}$$

$$\left.\begin{aligned} &= e(1-\tfrac{1}{2}\gamma^2+\tfrac{21}{64}\gamma^4-\tfrac{31}{128}\gamma^6)\cos(v-\omega)+\tfrac{3}{64}e\gamma^4(1-\tfrac{3}{2}\gamma^2)\cos 3(v-\omega) \\ &\quad +\tfrac{3}{8}e\gamma^2(1-\gamma^2+\tfrac{57}{64}\gamma^4)\cos(3v-\omega-2\Omega)+\tfrac{1}{8}e\gamma^2(1-\gamma^2+\tfrac{57}{64}\gamma^4)\cos(v-3\omega+2\Omega) \\ &\quad -\tfrac{3}{128}e\gamma^4(1-\tfrac{3}{2}\gamma^2)\cos(3v+\omega-4\Omega)+\tfrac{1}{128}e\gamma^4(1-\tfrac{3}{2}\gamma^2)\cos(v+3\omega-4\Omega) \\ &\quad +\tfrac{15}{128}e\gamma^4(1-\tfrac{3}{2}\gamma^2)\cos(5v-\omega-4\Omega)+\tfrac{3}{128}e\gamma^4(1-\tfrac{3}{2}\gamma^2)\cos(v-5\omega+4\Omega) \\ &\quad -\tfrac{5}{512}e\gamma^6\cos(5v+\omega-6\Omega)+\tfrac{1}{512}e\gamma^6\cos(v+5\omega-6\Omega)+\tfrac{35}{1024}e\gamma^6\cos(7v-\omega-6\Omega) \\ &\quad +\tfrac{15}{1024}e\gamma^6\cos(5v-3\omega-2\Omega)+\tfrac{9}{1024}e\gamma^6\cos(3v-5\omega+2\Omega) \\ &\quad +\tfrac{5}{1024}e\gamma^6\cos(v-7\omega+6\Omega) \end{aligned}\right\}, \tag{78}$$

$$\{e(1+\tfrac{1}{2}\gamma^2)\cos(v-\omega)-\tfrac{1}{2}e\gamma^2\cos(v+\omega-2\Omega)\}^2\cos^2\theta_0\{1+\gamma^2\sin^2(v-\Omega)\}^{-2}$$

$$\left.\begin{aligned} &= \tfrac{1}{2}e^2(1-\tfrac{1}{2}\gamma^2+\tfrac{3}{4}\gamma^4)+\tfrac{1}{2}e^2(1-\tfrac{1}{2}\gamma^2+\tfrac{1}{4}\gamma^4)\cos 2(v-\omega)+\tfrac{1}{8}e^2\gamma^2(1-\gamma^2)\cos 2(v-\Omega) \\ &\quad +\tfrac{1}{8}e^2\gamma^2(1-\gamma^2)\cos(2v-4\omega+2\Omega)+\tfrac{1}{32}e^2\gamma^4\cos(2v-6\omega+4\Omega) \\ &\quad +\tfrac{1}{16}e^2\gamma^4\cos 4(v-\omega)+\tfrac{3}{32}e^2\gamma^4\cos(6v-2\omega-4\Omega) \\ &\quad +\tfrac{1}{4}e^2\gamma^2(1-\gamma^2)\cos(4v-2\omega-2\Omega) \end{aligned}\right\}, \tag{79}$$

$$\{e(1+\tfrac{1}{2}\gamma^2)\cos(v-\omega)-\tfrac{1}{2}e\gamma^2\cos(v+\omega-2\Omega)\}^3\cos^3\theta_0\{1+\gamma^2\sin^2(v-\Omega)\}^{-\frac{5}{2}}$$

$$\left.\begin{aligned}
&=\tfrac{3}{4}e^3(1-\tfrac{1}{2}\gamma^2+\tfrac{11}{32}\gamma^4)\cos(v-\omega)+\tfrac{1}{4}e^3(1-\tfrac{1}{2}\gamma^2+\tfrac{15}{64}\gamma^4)\cos 3(v-\omega)\\
&\quad+\tfrac{15}{256}e^3\gamma^4\cos 5(v-\omega)+\tfrac{3}{16}e^3\gamma^2(1-\gamma^2)\cos(3v-\omega-2\Omega)\\
&\quad-\tfrac{3}{256}e^3\gamma^4\cos(3v+\omega-4\Omega)+\tfrac{1}{16}e^3\gamma^2(1-\gamma^2)\cos(v-3\omega+2\Omega)\\
&\quad+\tfrac{5}{32}e^3\gamma^2(1-\gamma^2)\cos(5v-3\omega-2\Omega)+\tfrac{3}{32}e^3\gamma^2(1-\gamma^2)\cos(3v-5\omega+2\Omega)\\
&\quad-\tfrac{3}{256}e^3\gamma^4\cos(3v+\omega-4\Omega)+\tfrac{1}{256}e^3\gamma^4\cos(v+3\omega-4\Omega)\\
&\quad+\tfrac{3}{512}e^3\gamma^4\cos(v-5\omega+4\Omega)+\tfrac{15}{512}e^3\gamma^4\cos(3v-7\omega+4\Omega)\\
&\quad+\tfrac{35}{512}e^3\gamma^4\cos(7nt-3\omega-4\Omega)
\end{aligned}\right\}\;; \quad (80)$$

$$\{e(1+\tfrac{1}{2}\gamma^2)\cos(v-\omega)-\tfrac{1}{2}e\gamma^2\cos(v+\omega-2\Omega)\}^4\cos^4\theta_0\{1+\gamma^2\sin(v-\Omega)\}^{-3}$$

$$\left.\begin{aligned}
&=\tfrac{3}{8}e^4(1-\tfrac{1}{2}\gamma^2)+\tfrac{1}{2}e^4(1-\tfrac{1}{2}\gamma^2)\cos 2(v-\omega)+\tfrac{1}{8}e^4(1-\tfrac{1}{2}\gamma^2)\cos 4(v-\omega)\\
&\quad+\tfrac{1}{16}e^4\gamma^2\cos 2(v-\Omega)+\tfrac{3}{32}e^4\gamma^2\cos(2v-4\omega+2\Omega)+\tfrac{3}{16}e^4\gamma^2\cos(4v-2\omega-2\Omega)\\
&\quad+\tfrac{3}{32}e^4\gamma^2\cos(6v-4\omega-2\Omega)+\tfrac{1}{16}e^4\gamma^2\cos(4v-6\omega+2\Omega)
\end{aligned}\right\}\;; \quad (81)$$

$$\{e(1+\tfrac{1}{2}\gamma^2)\cos(v-\omega)-\tfrac{1}{2}e\gamma^2\cos(v+\omega-2\Omega)\}^5\cos\theta_0{}^5\{1+\gamma^2\sin^2(v-\Omega)\}^{-\frac{7}{2}}$$

$$\left.\begin{aligned}
&=\tfrac{5}{8}e^5(1-\tfrac{1}{2}\gamma^2)\cos(v-\omega)+\tfrac{5}{16}e^5(1-\tfrac{1}{2}\gamma^2)\cos 3(v-\omega)\\
&\qquad+\tfrac{1}{16}e^5(1-\tfrac{1}{2}\gamma^2)\cos 5(v-\omega)+\tfrac{15}{128}e^5\gamma^2\cos(3v-\omega-2\Omega)\\
&\qquad+\tfrac{5}{128}e^5\gamma^2\cos(v-3\omega+2\Omega)+\tfrac{5}{132}e^5\gamma^2\cos(5v-3\omega-2\Omega)\\
&\qquad+\tfrac{3}{32}e^5\gamma^2\cos(3v-5\omega+2\Omega)+\tfrac{7}{128}e^5\gamma^2\cos(7nt-5\omega-2\Omega)\\
&\qquad+\tfrac{5}{128}e^5\gamma^2\cos(5v-7\omega+2\Omega)
\end{aligned}\right\}\;; \quad (82)$$

$$\left.\begin{aligned}
&\frac{\{e(1+\frac{1}{2}\gamma^2)\cos(v-\omega)-\frac{1}{2}e\gamma^2\cos(v+\omega-2\Omega)\}^6\cos^6\theta_0}{\{1+\gamma^2\sin^2(v-\Omega)\}^4}\\
&=\tfrac{5}{16}\gamma^6+\tfrac{15}{32}e^6\cos 2(v-\omega)+\tfrac{3}{16}e^6\cos 4(v-\omega)+\tfrac{1}{32}e^6\cos 6(v-\omega)
\end{aligned}\right\}\;; \quad (83)$$

$$\left.\begin{aligned}
&\frac{\{e(1+\frac{1}{2}\gamma^2)\cos(v-\omega)-\frac{1}{2}e\gamma^2\cos(v+\omega-2\Omega)\}^7\cos^7\theta_0}{\{1+\gamma^2\sin^2(v-\Omega)\}^{\frac{9}{2}}}\\
&=\tfrac{35}{64}e^7\cos(v-\omega)+\tfrac{21}{64}e^7\cos 3(v-\omega)+\tfrac{7}{64}e^7\cos 5(v-\omega)+\tfrac{1}{64}e^7\cos 7(v-\Omega)
\end{aligned}\right\}. \quad (84)$$

If we now multiply equation (78) by -2, (79) by $+3$, (80) by -4, (81) by $+5$, (82) by -6, (83) by $+7$, and (84) by -8; and then add the products to equation (76), we shall get the following development of equation (74):

$$\frac{\sqrt{\mu}}{a^{\frac{3}{2}}(1-e^2)^{\frac{3}{2}}\sqrt{1+\gamma^2}}\;\frac{dt}{dv}$$

$$\left.\begin{aligned}
&=1-\tfrac{1}{2}\gamma^2+\tfrac{3}{2}e^2+\tfrac{3}{8}\gamma^4+\tfrac{15}{8}e^4-\tfrac{5}{16}\gamma^6+\tfrac{35}{16}e^6-\tfrac{3}{4}e^2\gamma^2+\tfrac{9}{16}e^2\gamma^4-\tfrac{15}{16}e^4\gamma^2\\
&\quad-2e\left\{\begin{array}{l}1-\tfrac{1}{2}\gamma^2+\tfrac{3}{2}e^2+\tfrac{21}{64}\gamma^4+\tfrac{15}{8}e^4-\tfrac{3}{4}e^2\gamma^2\\ \quad-\tfrac{31}{128}\gamma^6+\tfrac{33}{64}e^2\gamma^4+\tfrac{35}{16}e^6-\tfrac{15}{16}e^4\gamma^2\end{array}\right\}\cos(v-\omega)\\
&\quad+\tfrac{3}{2}e^2\{1-\tfrac{1}{2}\gamma^2+\tfrac{5}{3}e^2+\tfrac{1}{4}\gamma^4+\tfrac{35}{16}e^4-\tfrac{5}{6}e^2\gamma^2\}\cos 2(v-\omega)\\
&\quad-\{e^3-\tfrac{1}{2}e^3\gamma^2+\tfrac{3}{32}e\gamma^4+\tfrac{15}{8}e^5+\tfrac{15}{64}e^3\gamma^4-\tfrac{9}{64}e\gamma^6+\tfrac{21}{8}e^7-\tfrac{15}{16}e^5\gamma^2\}\cos 3(v-\omega)\\
&\quad+\{\tfrac{5}{8}e^4-\tfrac{5}{16}e^4\gamma^2+\tfrac{3}{16}e^2\gamma^4+\tfrac{21}{16}e^6\}\cos 4(v-\omega)+\tfrac{7}{32}e^6\cos 6(v-\omega)\\
&\quad-\{\tfrac{3}{8}e^5+\tfrac{7}{8}e^7-\tfrac{8}{16}e^5\gamma^2+\tfrac{15}{64}e^3\gamma^4\}\cos 5(v-\omega)-\tfrac{1}{8}e^7\cos 7(v-\omega)\\
&\quad+\{\tfrac{1}{2}\gamma^2-\tfrac{1}{2}\gamma^4+\tfrac{3}{8}e^2\gamma^2+\tfrac{15}{32}\gamma^6-\tfrac{3}{8}e^2\gamma^4+\tfrac{5}{16}e^4\gamma^2\}\cos 2(v-\Omega)\\
&\quad+\tfrac{1}{8}\gamma^4\{1-\tfrac{3}{2}\gamma^2\}\cos 4(v-\Omega)+\tfrac{1}{32}\gamma^6\cos 6(v-\Omega)\\
&\quad-\{\tfrac{3}{4}e\gamma^2-\tfrac{3}{4}e\gamma^4+\tfrac{3}{4}e^3\gamma^2-\tfrac{3}{4}e^3\gamma^4+\tfrac{171}{256}e\gamma^6+\tfrac{45}{64}e^5\gamma^2\}\cos(3v-\omega-2\Omega)\\
&\quad-\{\tfrac{1}{4}e\gamma^2-\tfrac{1}{4}e\gamma^4+\tfrac{1}{4}e^3\gamma^2-\tfrac{1}{4}e^3\gamma^4+\tfrac{57}{256}e\gamma^6+\tfrac{15}{64}e^5\gamma^2\}\cos(v-3\omega+2\Omega)\\
&\quad+\{\tfrac{3}{64}e\gamma^4-\tfrac{9}{128}e\gamma^6+\tfrac{3}{64}e^3\gamma^4\}\cos(3v+\omega-4\Omega)\\
&\quad-\{\tfrac{1}{64}e\gamma^4-\tfrac{3}{128}e\gamma^6+\tfrac{1}{64}e^3\gamma^4\}\cos(v+3\omega-4\Omega)\\
&\quad-\{\tfrac{15}{64}e\gamma^4-\tfrac{45}{128}e\gamma^6+\tfrac{15}{128}e^3\gamma^4\}\cos(5v-\omega-4\Omega)\\
&\quad-\{\tfrac{3}{64}e\gamma^4-\tfrac{9}{128}e\gamma^6+\tfrac{3}{128}e^3\gamma^4\}\cos(v-5\omega+4\Omega)\\
&\quad+\tfrac{5}{256}e\gamma^6\cos(5v+\omega-6\Omega)-\tfrac{1}{256}e\gamma^6\cos(v+5\omega-6\Omega)\\
&\quad-\tfrac{35}{512}e\gamma^6\cos(7nt-\omega-6\Omega)-\tfrac{5}{512}e\gamma^6\cos(v-7\omega+6\Omega)\\
&\quad+\{\tfrac{3}{8}e^2\gamma^2-\tfrac{3}{8}e^2\gamma^4+\tfrac{15}{32}e^4\gamma^2\}\cos(2v-4\omega+2\Omega)\\
&\quad+\{\tfrac{3}{4}e^2\gamma^2-\tfrac{3}{4}e^2\gamma^4+\tfrac{15}{16}e^4\gamma^2\}\cos(4v-2\omega-2\Omega)\\
&\quad+\tfrac{3}{32}e^2\gamma^4\cos(2v-6\omega+4\Omega)+\tfrac{9}{32}e^2\gamma^4\cos(6v-2\omega-4\Omega)\\
&\quad-\{\tfrac{3}{8}e^3\gamma^2-\tfrac{3}{8}e^3\gamma^4+\tfrac{9}{512}e\gamma^6+\tfrac{9}{16}e^5\gamma^2\}\cos(3v-5\omega+2\Omega)\\
&\quad-\{\tfrac{5}{8}e^3\gamma^2-\tfrac{5}{8}e^3\gamma^4+\tfrac{15}{512}e\gamma^6+\tfrac{15}{16}e^5\gamma^2\}\cos(5v-3\omega-2\Omega)\\
&\quad+\tfrac{15}{32}e^4\gamma^2\cos(6v-4\omega-2\Omega)+\tfrac{5}{16}e^4\gamma^2\cos(4nt-6\omega+2\Omega)\\
&\quad-\tfrac{15}{128}e^3\gamma^4\cos(3nt-7\omega+4\Omega)-\tfrac{35}{128}e^3\gamma^4\cos(7v-3\omega-4\Omega)\\
&\quad-\tfrac{21}{64}e^5\gamma^2\cos(7nt-5\omega-2\Omega)-\tfrac{15}{64}e^5\gamma^2\cos(5v-7\omega+2\Omega)
\end{aligned}\right\}\qquad(85)$$

Now we have

$$(1-e^2)^{\frac{3}{2}}\sqrt{1+\gamma^2}=1-\tfrac{3}{2}e^2+\tfrac{1}{2}\gamma^2+\tfrac{3}{8}e^4-\tfrac{1}{8}\gamma^4-\tfrac{3}{4}e^2\gamma^2+\tfrac{3}{16}e^4\gamma^2+\tfrac{3}{16}e^2\gamma^4+\tfrac{1}{16}e^6+\tfrac{1}{16}\gamma^6. \quad (86)$$

Multiplying equation (85) by this value of $(1-e^2)^{\frac{3}{2}}\sqrt{1+\gamma^2}$, it becomes

$$\left.\begin{aligned}
\frac{\sqrt{\mu}}{a^{\frac{3}{2}}}\cdot\frac{dt}{dv} = 1 &- 2e\{1-\tfrac{3}{64}\gamma^4+\tfrac{3}{128}e^2\gamma^4+\tfrac{3}{64}\gamma^6\}\cos(v-\omega)\\
&+\tfrac{3}{2}e^2\{1+\tfrac{1}{6}e^2+\tfrac{1}{16}e^4-\tfrac{1}{8}\gamma^4\}\cos 2(v-\omega)\\
&-\{e^3+\tfrac{3}{8}e^5+\tfrac{3}{32}e\gamma^4+\tfrac{3}{16}e^7-\tfrac{9}{32}e^3\gamma^4-\tfrac{3}{32}e\gamma^6\}\cos 3(v-\omega)\\
&+\{\tfrac{5}{8}e^4+\tfrac{3}{16}e^2\gamma^4+\tfrac{3}{8}e^6\}\cos 4(v-\omega)+\tfrac{7}{32}e^6\cos 6(v-\omega)\\
&-\{\tfrac{3}{8}e^5+\tfrac{5}{16}e^7+\tfrac{15}{64}e^3\gamma^4\}\cos 5(v-\omega)-\tfrac{1}{8}e^7\cos 7(v-\omega)\\
&+\tfrac{1}{2}\gamma^2\{1-\tfrac{1}{2}\gamma^2-\tfrac{3}{4}e^2+\tfrac{5}{16}\gamma^4-\tfrac{1}{8}e^4+\tfrac{3}{8}e^2\gamma^2\}\cos 2(v-\Omega)\\
&+\tfrac{1}{8}\gamma^4\{1-\gamma^2-\tfrac{3}{2}e^2\}\cos 4(v-\Omega)+\tfrac{1}{32}\gamma^6\cos 6(v-\Omega)\\
&-\tfrac{3}{4}e\gamma^2\{1-\tfrac{1}{2}\gamma^2-\tfrac{1}{2}e^2+\tfrac{17}{64}\gamma^4-\tfrac{3}{16}e^4+\tfrac{1}{4}e^2\gamma^2\}\cos(3v-\omega-2\Omega)\\
&-\tfrac{1}{4}e\gamma^2\{1-\tfrac{1}{2}\gamma^2-\tfrac{1}{2}e^2+\tfrac{17}{64}\gamma^4-\tfrac{3}{16}e^4+\tfrac{1}{4}e^2\gamma^2\}\cos(v-3\omega+2\Omega)\\
&+\tfrac{3}{64}e\gamma^4\{1-\gamma^2-\tfrac{1}{2}e^2\}\cos(3v+\omega-4\Omega)-\tfrac{1}{64}e\gamma^4\{1-\gamma^2-\tfrac{1}{2}e^2\}\cos(v+3\omega-4\Omega)\\
&-\tfrac{15}{64}e\gamma^4\{1-\gamma^2-e^2\}\cos(5v-\omega-4\Omega)+\tfrac{5}{256}e\gamma^6\cos(5v+\omega-6\Omega)\\
&-\tfrac{3}{64}e\gamma^4\{1-\gamma^2-e^2\}\cos(v-5\omega+4\Omega)-\tfrac{1}{256}e\gamma^6\cos(v+5\omega-6\Omega)\\
&+\tfrac{3}{8}e^2\gamma^2\{1-\tfrac{1}{2}\gamma^2-\tfrac{1}{4}e^2\}\cos(2v-4\omega+2\Omega)-\tfrac{35}{512}e\gamma^6\cos(7v-\omega-6\Omega)\\
&+\tfrac{3}{4}e^2\gamma^2\{1-\tfrac{1}{2}\gamma^2-\tfrac{1}{4}e^2\}\cos(4v-2\omega-2\Omega)-\tfrac{5}{512}e\gamma^6\cos(v-7\omega+6\Omega)\\
&-\{\tfrac{3}{8}e^3\gamma^2-\tfrac{3}{16}e^3\gamma^4+\tfrac{9}{512}e\gamma^6\}\cos(3v-5\omega+2\Omega)+\tfrac{3}{32}e^2\gamma^4\cos(2v-6\omega+4\Omega)\\
&-\{\tfrac{5}{8}e^3\gamma^2-\tfrac{5}{16}e^3\gamma^4+\tfrac{15}{512}e\gamma^6\}\cos(5v-3\omega-2\Omega)+\tfrac{9}{32}e^2\gamma^4\cos(6v-2\omega-4\Omega)\\
&-\tfrac{15}{128}e^3\gamma^4\cos(3v-7\omega+4\Omega)-\tfrac{35}{128}e^3\gamma^4\cos(7v-3\omega-4\Omega)\\
&+\tfrac{15}{32}e^4\gamma^2\cos(6v-4\omega-2\Omega)+\tfrac{5}{16}e^4\gamma^2\cos(4v-6\omega+2\Omega)\\
&-\tfrac{21}{64}e^5\gamma^2\cos(7v-5\omega-2\Omega)-\tfrac{15}{64}e^5\gamma^2\cos(5v-7\omega+2\Omega)
\end{aligned}\right\}. \quad (87)$$

If we now put

$$n=\frac{\sqrt{\mu}}{a^{\frac{3}{2}}}, \quad (88)$$

equation (87) will give, by integration,

$$
\left.\begin{aligned}
nt = v &- 2e\left(1 - \tfrac{3}{64}\gamma^4 + \tfrac{3}{128}e^2\gamma^4 + \tfrac{3}{64}\gamma^6\right)\sin(v-\omega) \\
&+ \tfrac{3}{4}e^2\left(1 + \tfrac{1}{6}e^2 + \tfrac{1}{16}e^4 - \tfrac{1}{8}\gamma^4\right)\sin 2(v-\omega) \\
&- \left(\tfrac{1}{3}e^3 + \tfrac{1}{8}e^5 + \tfrac{1}{32}e\gamma^4 + \tfrac{1}{16}e^7 - \tfrac{3}{32}e^3\gamma^4 - \tfrac{1}{32}e\gamma^6\right)\sin 3(v-\omega) \\
&+ \left\{\tfrac{5}{32}e^4 + \tfrac{3}{32}e^6 + \tfrac{3}{64}e^2\gamma^4\right\}\sin 4(v-\omega) + \tfrac{7}{192}e^6\sin 6(v-\omega) \\
&- \left\{\tfrac{3}{40}e^5 + \tfrac{1}{16}e^7 + \tfrac{3}{64}e^3\gamma^4\right\}\sin 5(v-\omega) - \tfrac{1}{56}e^7\sin 7(v-\omega) \\
&+ \tfrac{1}{4}\gamma^2\left\{1 - \tfrac{1}{2}\gamma^2 - \tfrac{3}{4}e^2 + \tfrac{5}{16}\gamma^4 - \tfrac{1}{8}e^4 + \tfrac{3}{8}e^2\gamma^2\right\}\sin 2(v-\Omega) \\
&- \tfrac{1}{4}\gamma^2\left\{1 - \tfrac{1}{2}\gamma^2 - \tfrac{3}{4}e^2 + \tfrac{5}{16}\gamma^4 - \tfrac{1}{8}e^4 + \tfrac{3}{8}e^2\gamma^2\right\}\sin 2(\omega-\Omega) \\
&+ \tfrac{1}{32}\gamma^4\left\{1 - \gamma^2 - \tfrac{3}{2}e^2\right\}\sin 4(v-\Omega) + \tfrac{1}{192}\gamma^6\sin 6(v-\Omega) \\
&- \tfrac{1}{32}\gamma^4\left\{1 - \gamma^2 - \tfrac{3}{2}e^2\right\}\sin 4(\omega-\Omega) - \tfrac{1}{192}\gamma^6\sin 6(\omega-\Omega) \\
&- \tfrac{1}{4}e\gamma^2\left\{1 - \tfrac{1}{2}\gamma^2 - \tfrac{1}{2}e^2 + \tfrac{17}{64}\gamma^4 - \tfrac{3}{16}e^4 + \tfrac{1}{4}e^2\gamma^2\right\}\sin(3v-\omega-2\Omega) \\
&- \tfrac{1}{4}e\gamma^2\left\{1 - \tfrac{1}{2}\gamma^2 - \tfrac{1}{2}e^2 + \tfrac{17}{64}\gamma^4 - \tfrac{3}{16}e^4 + \tfrac{1}{4}e^2\gamma^2\right\}\sin(v-3\omega+2\Omega) \\
&+ \tfrac{1}{64}e\gamma^4\left\{1-\gamma^2-\tfrac{1}{2}e^2\right\}\sin(3v+\omega-4\Omega) - \tfrac{1}{64}e\gamma^4\left\{1-\gamma^2-\tfrac{1}{2}e^2\right\}\sin(v+3\omega-4\Omega) \\
&- \tfrac{3}{64}e\gamma^4\left\{1-\gamma^2-e^2\right\}\sin(5v-\omega-4\Omega) + \tfrac{1}{256}e\gamma^6\sin(5v+\omega-6\Omega) \\
&- \tfrac{3}{64}e\gamma^4\left\{1-\gamma^2-e^2\right\}\sin(v-5\omega+4\Omega) - \tfrac{1}{256}e\gamma^6\sin(v+5\omega-6\Omega) \\
&+ \tfrac{3}{16}e^2\gamma^2\left\{1-\tfrac{1}{2}\gamma^2-\tfrac{1}{4}e^2\right\}\sin(2v-4\omega+2\Omega) - \tfrac{5}{512}e\gamma^6\sin(7v-\omega-6\Omega) \\
&+ \tfrac{3}{16}e^2\gamma^2\left\{1-\tfrac{1}{2}\gamma^2-\tfrac{1}{4}e^2\right\}\sin(4v-2\omega-2\Omega) - \tfrac{5}{512}e\gamma^6\sin(v-7\omega+6\Omega) \\
&- \left\{\tfrac{1}{8}e^3\gamma^2 - \tfrac{1}{16}e^3\gamma^4 + \tfrac{3}{512}e\gamma^6\right\}\sin(3v-5\omega+2\Omega) + \tfrac{3}{64}e^2\gamma^4\sin(2v-6\omega+4\Omega) \\
&- \left\{\tfrac{1}{8}e^3\gamma^2 - \tfrac{1}{16}e^3\gamma^4 + \tfrac{3}{512}e\gamma^6\right\}\sin(5v-3\omega-2\Omega) + \tfrac{3}{64}e^2\gamma^4\sin(6v-2\omega-4\Omega) \\
&- \tfrac{5}{128}e^3\gamma^4\sin(3v-7\omega+4\Omega) - \tfrac{5}{128}e^3\gamma^4\sin(7v-3\omega-4\Omega) \\
&+ \tfrac{5}{64}e^4\gamma^2\sin(6v-4\omega-2\Omega) + \tfrac{5}{64}e^4\gamma^2\sin(4v-6\omega+2\Omega) \\
&- \tfrac{3}{64}e^5\gamma^2\sin(7v-5\omega-2\Omega) - \tfrac{3}{64}e^5\gamma^2\sin(5v-7\omega+2\Omega)
\end{aligned}\right\} \quad (89)
$$

In this equation nt denotes the mean longitude of the moon, and v the true longitude, and the constant introduced by the integrations has been made to satisfy the condition that the *mean* and *true* longitudes shall be equal to each other at the extremities of the transverse axis By inverting this series we may obtain the value of v in terms of nt

5 For this purpose we shall observe that if we change the sign of all the terms of the second member except the first, and at the same time change v into

nt in those terms, and calling the sum of the terms thus changed $f(nt)$, we shall have, according to the theorem of LA GRANGE,

$$\left.\begin{aligned} v = nt + f(nt) + \tfrac{1}{2}\frac{df(nt)^2}{ndt} + \tfrac{1}{6}\frac{d^2f(nt)^3}{n^2dt^2} + \tfrac{1}{24}\frac{d^3f(nt)^4}{n^3dt^3} \\ + \tfrac{1}{120}\frac{d^4f(nt)^5}{n^4dt^4} + \tfrac{1}{720}\frac{d^5f(nt)^6}{n^5dt^5} + \tfrac{1}{5040}\frac{d^6f(nt)^7}{n^6dt^6} \end{aligned}\right\} \tag{90}$$

We shall therefore have

$$\left.\begin{aligned}
f(nt) = {} & 2e\,\{1 - \tfrac{3}{64}\gamma^4 + \tfrac{3}{128}e^2\gamma^4 + \tfrac{3}{64}\gamma^6\}\sin(nt-\omega) \\
& - \tfrac{3}{4}e^2\{1 + \tfrac{1}{6}e^2 + \tfrac{1}{16}e^4 - \tfrac{1}{8}\gamma^4\}\sin 2(nt-\omega) \\
& + \{\tfrac{1}{3}e^3 + \tfrac{1}{8}e^5 + \tfrac{1}{32}e\gamma^4 + \tfrac{1}{16}e^7 - \tfrac{3}{32}e^3\gamma^4 - \tfrac{1}{32}e\gamma^6\}\sin 3(nt-\omega) \\
& - \{\tfrac{5}{32}e^4 + \tfrac{3}{32}e^6 + \tfrac{3}{64}e^2\gamma^4\}\sin 4(nt-\omega) - \tfrac{7}{192}e^6\sin 6(nt-\omega) \\
& + \{\tfrac{3}{40}e^5 + \tfrac{1}{16}e^7 + \tfrac{3}{64}e^3\gamma^4\}\sin 5(nt-\omega) + \tfrac{1}{56}e^7\sin^7(nt-\omega) \\
& - \tfrac{1}{4}\gamma^2\{1 - \tfrac{1}{2}\gamma^2 - \tfrac{3}{4}e^2 + \tfrac{5}{16}\gamma^4 - \tfrac{1}{8}e^4 + \tfrac{3}{8}e^2\gamma^2\}\sin 2(nt-\Omega) \\
& + \tfrac{1}{4}\gamma^2\{1 - \tfrac{1}{2}\gamma^2 - \tfrac{3}{4}e^2 + \tfrac{5}{16}\gamma^4 - \tfrac{1}{8}e^4 + \tfrac{3}{8}e^2\gamma^2\}\sin 2(\omega-\Omega) \\
& - \tfrac{1}{32}\gamma^4\{1 - \gamma^2 - \tfrac{3}{2}e^2\}\sin 4(nt-\Omega) - \tfrac{1}{192}\gamma^6\sin 6(nt-\Omega) \\
& + \tfrac{1}{32}\gamma^4\{1 - \gamma^2 - \tfrac{3}{2}e^2\}\sin 4(\omega-\Omega) + \tfrac{1}{192}\gamma^6\sin 6(\omega-\Omega) \\
& + \tfrac{1}{4}e\gamma^2\{1 - \tfrac{1}{2}\gamma^2 - \tfrac{1}{2}e^2 + \tfrac{17}{64}\gamma^4 - \tfrac{3}{16}e^4 + \tfrac{1}{4}e^2\gamma^2\}\sin(3nt-\omega-2\Omega) \\
& + \tfrac{1}{4}e\gamma^2\{1 - \tfrac{1}{2}\gamma^2 - \tfrac{1}{2}e^2 + \tfrac{17}{64}\gamma^4 - \tfrac{3}{16}e^4 + \tfrac{1}{4}e^2\gamma^2\}\sin(nt-3\omega+2\Omega) \\
& - \tfrac{1}{64}e\gamma^4\{1-\gamma^2-\tfrac{1}{2}e^2\}\sin(3nt+\omega-4\Omega) + \tfrac{1}{64}e\gamma^4\{1-\gamma^2-\tfrac{1}{2}e^2\}\sin(nt+3\omega-4\Omega) \\
& + \tfrac{3}{64}e\gamma^4\{1 - \gamma^2 - e^2\}\sin(5nt-\omega-4\Omega) - \tfrac{1}{256}e\gamma^6\sin(5nt+\omega-6\Omega) \\
& + \tfrac{3}{64}e\gamma^4\{1 - \gamma^2 - e^2)\sin(nt-5\omega+4\Omega) + \tfrac{1}{256}e\gamma^6\sin(nt+5\omega-6\Omega) \\
& - \tfrac{3}{16}e^2\gamma^2\{1 - \tfrac{1}{2}\gamma^2 - \tfrac{1}{4}e^2\}\sin(2nt-4\omega+2\Omega) + \tfrac{5}{512}e\gamma^6\sin(7nt-\omega-6\Omega) \\
& - \tfrac{3}{16}e^2\gamma^2\{1 - \tfrac{1}{2}\gamma^2 - \tfrac{1}{4}e^2\}\sin(4nt-2\omega-2\Omega) + \tfrac{5}{512}e\gamma^6\sin(nt-7\omega+6\Omega) \\
& + \{\tfrac{1}{8}e^3\gamma^2 - \tfrac{1}{16}e^3\gamma^4 + \tfrac{3}{512}e\gamma^6\}\sin(3nt-5\omega+2\Omega) - \tfrac{3}{64}e^2\gamma^4\sin(2nt-6\omega+4\Omega) \\
& + \{\tfrac{1}{8}e^3\gamma^2 - \tfrac{1}{16}e^3\gamma^4 + \tfrac{3}{512}e\gamma^6\}\sin(5nt-3\omega+2\Omega) - \tfrac{3}{64}e^2\gamma^4\sin(6nt-2\omega-4\Omega) \\
& + \tfrac{5}{128}e^3\gamma^4\sin(3nt-7\omega+4\Omega) + \tfrac{5}{128}e^3\gamma^4\sin(7nt-3\omega-4\Omega) \\
& - \tfrac{5}{64}e^4\gamma^2\sin(6nt-4\omega-2\Omega) - \tfrac{5}{64}e^4\gamma^2\sin(4nt-6\omega+2\Omega) \\
& + \tfrac{3}{64}e^5\gamma^2\sin(7nt-5\omega-2\Omega) + \tfrac{3}{64}e^5\gamma^2\sin(5nt-7\omega+2\Omega)
\end{aligned}\right\}. \tag{91}$$

From this equation we get the following values of the different powers of $f(nt)$, which are all correct to terms of the seventh order

$$\left.\begin{aligned}
f(nt)^2 = {} & 2e^2 + \tfrac{9}{32}e^4 + \tfrac{1}{16}\gamma^4 + \tfrac{43}{288}e^6 - \tfrac{1}{16}\gamma^6 - \tfrac{7}{32}e^2\gamma^4 \\
& -\{\tfrac{3}{2}e^3 + \tfrac{1}{2}e^5 + \tfrac{1}{8}e\gamma^4 + \tfrac{9}{32}e^7 - \tfrac{19}{64}e^3\gamma^4 - \tfrac{1}{8}e\gamma^6\}\cos(nt-\omega) \\
& -\{2e^2 - \tfrac{2}{3}e^4 + \tfrac{1}{16}\gamma^4 - \tfrac{47}{128}e^6 - \tfrac{7}{16}e^2\gamma^4 - \tfrac{1}{16}\gamma^6\}\cos 2(nt-\omega) \\
& +\{\tfrac{3}{2}e^3 - \tfrac{1}{16}e^5 + \tfrac{1}{8}e\gamma^4 - \tfrac{3}{20}e^7 - \tfrac{73}{128}e^3\gamma^4 - \tfrac{1}{8}\gamma^6\}\cos 3(nt-\omega) \\
& -\{\tfrac{91}{96}e^4 + \tfrac{31}{160}e^6 + \tfrac{7}{32}e^2\gamma^4\}\cos 4(nt-\omega) - \tfrac{1859}{5760}e^6\cos 6(nt-\omega) \\
& +\{\tfrac{9}{16}e^5 + \tfrac{1}{4}e^7 + \tfrac{35}{128}e^3\gamma^4\}\cos 5(nt-\omega) + \tfrac{29}{160}e^7\cos 7(nt-\omega) \\
& +\{\tfrac{11}{16}e^2\gamma^2 - \tfrac{11}{32}e^2\gamma^4 - \tfrac{13}{96}e^4\gamma^2 + \tfrac{1}{128}\gamma^6\}\{\cos 2(nt-\Omega) + \cos 2(\omega-\Omega)\} \\
& -\{\tfrac{1}{32}\gamma^4 - \tfrac{1}{32}\gamma^6 - \tfrac{7}{32}e^2\gamma^4\}\{\cos 4(nt-\Omega) + \cos 4(\omega-\Omega)\} \\
& -\tfrac{1}{128}\gamma^6\{\cos 6(nt-\Omega) + \cos 6(\omega-\Omega)\} - \tfrac{1}{128}e\gamma^6\cos(3nt+3\omega-6\Omega) \\
& -\{e\gamma^2 - \tfrac{1}{2}e\gamma^4 + \tfrac{35}{128}e\gamma^6 - \tfrac{1}{8}e^5\gamma^2 - \tfrac{3}{8}e^3\gamma^2 + \tfrac{3}{16}e^3\gamma^4\}\cos(nt+\omega-2\Omega) \\
& +\{\tfrac{1}{2}e\gamma^2 - \tfrac{1}{4}e\gamma^4 - \tfrac{5}{6}e^3\gamma^2 + \tfrac{5}{12}e^3\gamma^4 - \tfrac{9}{128}e^5\gamma^2 + \tfrac{7}{64}e\gamma^6\}\cos(nt-3\omega+2\Omega) \\
& +\{\tfrac{1}{2}e\gamma^2 - \tfrac{1}{4}e\gamma^4 - \tfrac{5}{6}e^3\gamma^2 + \tfrac{5}{12}e^3\gamma^4 - \tfrac{9}{128}e^5\gamma^2 + \tfrac{7}{64}e\gamma^6\}\cos(3nt-\omega-2\Omega) \\
& +\{\tfrac{1}{8}e\gamma^4 - \tfrac{1}{8}e\gamma^6 - \tfrac{79}{256}e^3\gamma^4\}\{\cos(nt-5\omega+4\Omega) + \cos(5nt-\omega-4\Omega)\} \\
& -\{\tfrac{11}{16}e\gamma^2 - \tfrac{11}{32}e^2\gamma^4 - \tfrac{83}{128}e^4\gamma^2 + \tfrac{1}{128}\gamma^6\}\{\cos(2nt-4\omega+2\Omega) + \cos(4nt-2\omega-2\Omega)\} \\
& +\{\tfrac{31}{48}e^3\gamma - \tfrac{31}{96}e^3\gamma^4 - \tfrac{29}{80}e^5\gamma^2 + \tfrac{7}{256}e\gamma^6\}\{\cos(3nt-5\omega+2\Omega) + \cos(5nt-3\omega-2\Omega)\} \\
& -\{\tfrac{1}{8}e\gamma^4 - \tfrac{1}{8}e\gamma^6 - \tfrac{35}{384}e^3\gamma^4\}\{\cos(3nt+\omega-4\Omega) + \cos(nt+3\omega-4\Omega) \\
& +\{\tfrac{1}{16}\gamma^4 - \tfrac{1}{16}\gamma^6 - \tfrac{3}{64}e^2\gamma^4\}\cos(2nt+2\omega-4\Omega) \\
& -\tfrac{25}{128}e^2\gamma^4\{\cos(6nt-2\omega-4\Omega) + \cos(2nt-6\omega+4\Omega)\} \\
& -\tfrac{197}{384}e^4\gamma^2\{\cos(6nt-4\omega-2\Omega) + \cos(4nt-6\omega+2\Omega)\} \\
& +\tfrac{23}{768}e\gamma^6\{\cos(nt-7\omega+6\Omega) + \cos(7nt-\omega-6\Omega)\} \\
& -\tfrac{5}{192}e\gamma^6\{\cos(nt+5\omega-6\Omega) + \cos(5nt+\omega-6\Omega)\} \\
& +\tfrac{167}{768}e^3\gamma^4\{\cos(7nt-3\omega-4\Omega) + \cos(3nt-7\omega+4\Omega)\} \\
& +\tfrac{287}{640}e^5\gamma^2\{\cos(7nt-5\omega-2\Omega) + \cos(5nt-7\omega+2\Omega)\} \\
& +\tfrac{1}{128}\gamma^6\{\cos(4nt+2\omega-6\Omega) + \cos(2nt+4\omega-6\Omega)\}
\end{aligned}\right\} \qquad (92)$$

$$
\left.
\begin{aligned}
f(nt)^3 = {} & \{6e^3 + \tfrac{11}{16}e^5 + \tfrac{9}{16}e\gamma^4 + \tfrac{119}{384}e^7 - \tfrac{9}{16}e\gamma^6 - \tfrac{23}{16}e^3\gamma^4\} \sin(nt - \omega) \\
& - \{\tfrac{9}{2}e^4 + \tfrac{345}{256}e^6 + \tfrac{57}{64}e^2\gamma^4\} \sin 2(nt - \omega) \\
& - \{2e^3 - \tfrac{91}{32}e^5 + \tfrac{3}{16}e\gamma^4 - \tfrac{921}{640}e^7 - \tfrac{3}{16}e\gamma^6 - \tfrac{109}{64}e^3\gamma^4\} \sin 3(nt - \omega) \\
& + \{\tfrac{9}{4}e^4 - \tfrac{21}{16}e^6 + \tfrac{57}{128}e^2\gamma^4\} \sin 4(nt - \omega) + \tfrac{339}{256}e^6 \sin 6(nt - \omega) \\
& - \{\tfrac{59}{32}e^5 - \tfrac{599}{1920}e^7 + \tfrac{47}{64}e^3\gamma^4\} \sin 5(nt - \omega) - \tfrac{1697}{1920}e^7 \sin 7(nt - \omega) \\
& + \{\tfrac{9}{4}e^2\gamma^2 - \tfrac{9}{8}e^2\gamma^4 - \tfrac{149}{128}e^4\gamma^2 + \tfrac{9}{256}\gamma^6\} \{\sin 2(\omega - \Omega) - \sin 2(nt - \Omega)\} \\
& + \{\tfrac{9}{32}e\gamma^4 - \tfrac{9}{32}e\gamma^6 - \tfrac{25}{64}e^3\gamma^4\} \{\sin(3nt + \omega - 4\Omega) - \sin(nt + 3\omega - 4\Omega)\} \\
& - \{\tfrac{3}{32}e\gamma^4 - \tfrac{3}{32}e\gamma^6 - \tfrac{85}{64}e^3\gamma^4\} \{\sin(nt - 5\omega + 4\Omega) + \sin(5nt - \omega - 4\Omega)\} \\
& + \{\tfrac{21}{8}e^3\gamma^2 - \tfrac{21}{16}e^3\gamma^4 - \tfrac{299}{256}e^5\gamma^2 + \tfrac{15}{128}e\gamma^6\} \sin(3nt - \omega - 2\Omega) \\
& + \{\tfrac{21}{8}e^3\gamma^2 - \tfrac{21}{16}e^3\gamma^4 - \tfrac{299}{256}e^5\gamma^2 + \tfrac{15}{128}e\gamma^6\} \sin(nt - 3\omega + 2\Omega) \\
& - \{\tfrac{21}{16}e^3\gamma^2 - \tfrac{21}{32}e^3\gamma^4 - \tfrac{157}{64}e^5\gamma^2 + \tfrac{15}{256}e\gamma^6\} \sin(5nt - 3\omega - 2\Omega) \\
& - \{\tfrac{21}{16}e^3\gamma^2 - \tfrac{21}{32}e^3\gamma^4 - \tfrac{157}{64}e^5\gamma^2 + \tfrac{15}{256}e\gamma^6\} \sin(3nt - 5\omega + 2\Omega) \\
& + \{\tfrac{3}{4}e^2\gamma^2 - \tfrac{3}{8}e^2\gamma^4 - \tfrac{731}{256}e^4\gamma^2 + \tfrac{3}{256}\gamma^6\} \sin(4nt - 2\omega - 2\Omega) \\
& + \{\tfrac{3}{4}e^2\gamma^2 - \tfrac{3}{8}e^2\gamma^4 - \tfrac{731}{256}e^4\gamma^2 + \tfrac{3}{256}\gamma^6\} \sin(2nt - 4\omega + 2\Omega) \\
& + \tfrac{81}{256}e^2\gamma^4 \{\sin(2nt - 6\omega + 4\Omega) + \sin(6nt - 2\omega - 4\Omega)\} \\
& + \tfrac{379}{256}e^4\gamma^2 \{\sin(4nt - 6\omega + 2\Omega) + \sin(6nt - 4\omega - 2\Omega)\} \\
& + \tfrac{81}{128}e^2\gamma^4 \{\sin 4(\omega - \Omega) - \sin 4(nt - \Omega)\} \\
& - \tfrac{1}{256}\gamma^6 \{\sin 6(\omega - \Omega) - \sin 6(nt - \Omega)\} \\
& + \tfrac{3}{256}\gamma^6 \{\sin(2nt + 4\omega - 6\Omega) - \sin(4nt + 2\omega - 6\Omega)\} \\
& - \tfrac{17}{32}e^3\gamma^4 \{\sin(7nt - 3\omega - 4\Omega) + \sin(3nt - 7\omega + 4\Omega)\} \\
& - \tfrac{349}{256}e^5\gamma^2 \{\sin(5nt - 7\omega + 2\Omega) + \sin(7nt - 5\omega - 2\Omega)\} \\
& + \tfrac{9}{128}e\gamma^6 \{\sin(5nt + \omega - 6\Omega) - \sin(nt + 5\omega - 6\Omega)\} \\
& - \tfrac{9}{256}e\gamma^6 \{\sin(7nt - \omega - 6\Omega) + \sin(nt - 7\omega + 6\Omega)\}
\end{aligned}
\right\} \quad (93)
$$

$$
\left.
\begin{aligned}
f(nt)^4 = {} & 6e^4 + \tfrac{49}{24}e^6 + \tfrac{9}{8}e^2\gamma^4 \\
& - \{6e^5 + \tfrac{201}{64}e^7 + \tfrac{33}{16}e^3\gamma^4\} \cos(nt - \omega) \\
& - \{8e^4 - \tfrac{37}{16}e^6 + \tfrac{3}{2}e^2\gamma^4\} \cos 2(nt - \omega) \\
& + \{9e^5 + \tfrac{57}{64}e^7 + \tfrac{99}{32}e^3\gamma^4\} \cos 3(nt - \omega) \\
& + \{2e^4 - \tfrac{59}{8}e^6 + \tfrac{3}{8}e^2\gamma^4\} \cos 4(nt - \omega) + \tfrac{145}{48}e^6 \cos 6(nt - \omega) \\
& - \{3e^5 - \tfrac{307}{64}e^7 + \tfrac{37}{32}e^3\gamma^4\} \cos 5(nt - \omega) - \tfrac{163}{64}e^7 \cos 7(nt - \omega) \\
& - \{6e^3\gamma^2 - 3e^3\gamma^4 + \tfrac{11}{16}e^5\gamma^2 + \tfrac{9}{32}e\gamma^6\} \cos(nt + \omega - 2\Omega) \\
& + \{4e^3\gamma^2 - 2e^3\gamma^4 - \tfrac{189}{64}e^5\gamma^2 + \tfrac{3}{16}e\gamma^6\} \cos(3nt - \omega - 2\Omega) \\
& + \{4e^3\gamma^2 - 2e^3\gamma^4 - \tfrac{189}{64}e^5\gamma^2 + \tfrac{3}{16}e\gamma^6\} \cos(nt - 3\omega + 2\Omega) \\
& - \{e^3\gamma^2 - \tfrac{1}{2}e^3\gamma^4 - \tfrac{171}{32}e^5\gamma^2 + \tfrac{3}{64}e\gamma^6\} \cos(3nt - 5\omega + 2\Omega) \\
& - \{e^3\gamma^2 - \tfrac{1}{2}e^3\gamma^4 - \tfrac{171}{32}e^5\gamma^2 + \tfrac{3}{64}e\gamma^6\} \cos(5nt - 3\omega - 2\Omega) \\
& + \tfrac{9}{8}e^2\gamma^4 \cos(2nt + 2\omega - 4\Omega) - \tfrac{3}{32}e\gamma^6 \cos(3nt + 3\omega - 6\Omega) \\
& - \tfrac{1}{64}e\gamma^6 \{\cos(7nt - \omega - 6\Omega) + \cos(nt - 7\omega + 6\Omega)\} \\
& - \tfrac{131}{64}e^5\gamma^2 \{\cos(5nt - 7\omega + 2\Omega) + \cos(7nt - 5\omega - 2\Omega)\} \\
& - \tfrac{41}{32}e^3\gamma^4 \{\cos(3nt + \omega - 4\Omega) + \cos(nt + 3\omega - 4\Omega)\} \\
& - \tfrac{41}{64}e^3\gamma^4 \{\cos(3nt - 7\omega + 4\Omega) + \cos(7nt - 3\omega - 4\Omega)\} \\
& + \tfrac{3}{16}e^2\gamma^4 \{\cos(2nt - 6\omega + 4\Omega) + \cos(6nt - 2\omega - 4\Omega)\} \\
& - \tfrac{51}{8}e^4\gamma^2 \{\cos(2nt - 4\omega + 2\Omega) + \cos(4nt - 2\omega - 2\Omega)\} \\
& + \tfrac{17}{8}e^4\gamma^2 \{\cos(4nt - 6\omega + 2\Omega) + \cos(6nt - 4\omega - 2\Omega)\} \\
& + \tfrac{123}{64}e^3\gamma^4 \{\cos(nt - 5\omega + 4\Omega) + \cos(5nt - \omega - 4\Omega)\} \\
& + \tfrac{1}{16}e\gamma^6 \{\cos(nt + 5\omega - 6\Omega) + \cos(5nt + \omega - 6\Omega)\} \\
& - \tfrac{3}{4}e^2\gamma^4 \{\cos 4(\omega - \Omega) + \cos 4(nt - \Omega)\} \\
& + \tfrac{17}{4}e^4\gamma^2 \{\cos 2(\omega - \Omega) + \cos 2(nt - \Omega)\}
\end{aligned}
\right\}, \qquad (94)
$$

$$\left.\begin{aligned}
f(nt)^5 = {} & \{20e^5 + \tfrac{275}{48}e^7 + \tfrac{25}{4}e^3\gamma^4\} \sin(nt - \omega) - \tfrac{75}{4}e^6 \sin 2(nt - \omega) \\
& - \{10e^5 - \tfrac{205}{16}e^7 + \tfrac{25}{8}e^3\gamma^4\} \sin 3(nt - \omega) + 15e^6 \cos 4(nt - \omega) \\
& + \{2e^5 - \tfrac{725}{48}e^7 + \tfrac{5}{8}e^3\gamma^4\} \sin 5(nt - \omega) - \tfrac{15}{4}e^6 \sin 6(nt - \omega) \\
& + \tfrac{215}{48}e^7 \sin 7(nt - \omega) + \tfrac{25}{2}e^4\gamma^2 \{\sin 2(\omega - \Omega) - \sin 2(nt - \Omega)\} \\
& + \tfrac{25}{4}e^4\gamma^2 \{\sin(2nt - 4\omega + 2\Omega) + \sin(4nt - 2\omega - 2\Omega)\} \\
& - \tfrac{5}{4}e^4\gamma^2 \{\sin(4nt - 6\omega + 2\Omega) + \sin(6nt - 4\omega - 2\Omega)\} \\
& + \tfrac{25}{8}e^5\gamma^2 \{\sin(5nt - 7\omega + 2\Omega) + \sin(7nt - 5\omega - 2\Omega)\} \\
& + \tfrac{125}{8}e^5\gamma^2 \{\sin(nt - 3\omega + 2\Omega) + \sin(3nt - \omega - 2\Omega)\} \\
& - \tfrac{25}{2}e^5\gamma^2 \{\sin(3nt - 5\omega + 2\Omega) + \sin(5nt - 3\omega - 2\Omega)\} \\
& + \tfrac{25}{8}e^3\gamma^4 \{\sin(3nt + \omega - 4\Omega) - \sin(nt + 3\omega - 4\Omega)\} \\
& - \tfrac{25}{16}e^3\gamma^4 \{\sin(nt - 5\omega + 4\Omega) + \sin(5nt - \omega - 4\Omega)\} \\
& + \tfrac{5}{16}e^3\gamma^4 \{\sin(3nt - 7\omega + 4\Omega) + \sin(7nt - 3\omega - 4\Omega)\}
\end{aligned}\right\}; \qquad (95)$$

$$\left.\begin{aligned}
f(nt)^6 = {} & 20e^6 - \tfrac{45}{2}e^7 \cos(nt - \omega) - 30e^6 \cos 2(nt - \omega) \\
& + \tfrac{81}{2}e^7 \cos 3(nt - \omega) + 12e^6 \cos 4(nt - \omega) - \tfrac{45}{2}e^7 \cos 5(nt - \omega) \\
& - 2e^6 \cos 6(nt - \omega) + \tfrac{9}{2}e^7 \cos 7(nt - \omega) - 30e^5\gamma^2 \cos(nt + \omega - 2\Omega) \\
& + \tfrac{45}{2}e^5\gamma^2 \{\cos(3nt - \omega - 2\Omega) + \cos(nt - 3\omega + 2\Omega)\} \\
& - 9e^5\gamma^2 \{\cos(5nt - 3\omega - 2\Omega) + \cos(3nt - 5\omega + 2\Omega)\} \\
& + \tfrac{9}{2}e^5\gamma^2 \{\cos(7nt - 5\omega - 2\Omega) + \cos(5nt - 7\omega + 2\Omega)\}
\end{aligned}\right\}; \qquad (96)$$

$$\left.\begin{aligned}
f(nt)^7 = {} & 70e^7 \sin(nt - \omega) - 42e^7 \sin 3(nt - \omega) \\
& + 14e^5 \sin 5(nt - \omega) - 2e^7 \sin 7(nt - \omega)
\end{aligned}\right\}. \qquad (97)$$

$$
\frac{1}{2}\frac{df(nt)^2}{ndt} =
$$

$$
\left.\begin{aligned}
&\{\tfrac{3}{4}e^3+\tfrac{1}{4}e^5+\tfrac{1}{16}e\gamma^4+\tfrac{9}{64}e^7-\tfrac{1}{16}e\gamma^6-\tfrac{19}{128}e^3\gamma^4\}\sin(nt-\omega)\\
&+\{2e^2-\tfrac{2}{3}e^4+\tfrac{1}{16}\gamma^4-\tfrac{1}{16}\gamma^6-\tfrac{47}{128}e^6-\tfrac{7}{16}e\gamma^4\}\sin 2(nt-\omega)\\
&-\{\tfrac{9}{4}e^3-\tfrac{3}{32}e^5+\tfrac{3}{16}e\gamma^4-\tfrac{9}{40}e^7-\tfrac{3}{16}e\gamma^6-\tfrac{219}{256}e^3\gamma^4\}\sin 3(nt-\omega)\\
&+\{\tfrac{91}{48}e^4+\tfrac{31}{80}e^6+\tfrac{7}{16}e\gamma^4\}\sin 4(nt-\omega)+\tfrac{1859}{1920}e^6\sin 6(nt-\omega)\\
&-\{\tfrac{45}{32}e^5+\tfrac{5}{8}e^7+\tfrac{175}{256}e^3\gamma^4\}\sin 5(nt-\omega)-\tfrac{203}{320}e^7\sin 7(nt-\omega)\\
&-\{\tfrac{11}{16}e^2\gamma^2-\tfrac{11}{32}e^2\gamma^4-\tfrac{13}{96}e^4\gamma^2+\tfrac{1}{128}\gamma^6\}\sin 2(nt-\Omega)\\
&+\{\tfrac{1}{16}\gamma^4-\tfrac{1}{16}\gamma^6-\tfrac{7}{16}e^2\gamma^4\}\sin 4(nt-\Omega)+\tfrac{3}{128}\gamma^6\sin 6(nt-\Omega)\\
&+\{\tfrac{1}{2}e\gamma^2-\tfrac{1}{4}e\gamma^4+\tfrac{35}{256}e\gamma^6-\tfrac{1}{16}e^5\gamma^2-\tfrac{3}{16}e^3\gamma^2+\tfrac{3}{32}e^3\gamma^4\}\sin(nt+\omega-2\Omega)\\
&-\{\tfrac{1}{4}e\gamma^2-\tfrac{1}{8}e\gamma^4-\tfrac{5}{12}e^3\gamma^2+\tfrac{5}{24}e^3\gamma^4-\tfrac{9}{256}e^5\gamma^2+\tfrac{7}{128}e\gamma^6\}\sin(nt-3\omega+2\Omega)\\
&-\{\tfrac{3}{4}e\gamma^2-\tfrac{3}{8}e\gamma^4-\tfrac{5}{4}e^3\gamma^2+\tfrac{5}{8}e^3\gamma^4-\tfrac{27}{256}e^5\gamma^2+\tfrac{21}{128}e\gamma^6\}\sin(3nt-\omega-2\Omega)\\
&+\{\tfrac{11}{16}e^2\gamma^2-\tfrac{11}{32}e^2\gamma^4-\tfrac{83}{128}e^4\gamma^2+\tfrac{1}{128}\gamma^6\}\sin(2nt-4\omega+2\Omega)\\
&+\{\tfrac{11}{8}e^2\gamma^2-\tfrac{11}{16}e^2\gamma^4-\tfrac{83}{64}e^4\gamma^2+\tfrac{1}{64}\gamma^6\}\sin(4nt-2\omega-2\Omega)\\
&-\{\tfrac{93}{96}e^3\gamma^2-\tfrac{93}{192}e^3\gamma^4-\tfrac{87}{160}e^5\gamma^2+\tfrac{21}{512}e\gamma^6\}\sin(3nt-5\omega+2\Omega)\\
&-\{\tfrac{155}{96}e^3\gamma^2-\tfrac{155}{192}e^3\gamma^4-\tfrac{29}{32}e^5\gamma^2+\tfrac{35}{512}e\gamma^6\}\sin(5nt-3\omega-2\Omega)\\
&-\{\tfrac{1}{16}\gamma^4-\tfrac{1}{16}\gamma^6-\tfrac{3}{64}e^2\gamma^4\}\sin(2nt+2\omega-4\Omega)+\tfrac{3}{256}e\gamma^6\sin(3nt+3\omega-6\Omega)\\
&-\{\tfrac{1}{16}e\gamma^4-\tfrac{1}{16}e\gamma^6-\tfrac{79}{512}e^3\gamma^4\}\sin(nt-5\omega+4\Omega)\\
&-\{\tfrac{5}{16}e\gamma^4-\tfrac{5}{16}e\gamma^6-\tfrac{395}{512}e^3\gamma^4\}\sin(5nt-\omega-4\Omega)\\
&+\{\tfrac{3}{16}e\gamma^4-\tfrac{3}{16}e\gamma^6-\tfrac{35}{256}e^3\gamma^4\}\sin(3nt+\omega-4\Omega)\\
&+\{\tfrac{1}{16}e\gamma^4-\tfrac{1}{16}e\gamma^6-\tfrac{35}{768}e^3\gamma^4\}\sin(nt+3\omega-4\Omega)\\
&+\tfrac{75}{128}e^2\gamma^4\sin(6nt-2\omega-4\Omega)+\tfrac{25}{128}e^2\gamma^4\sin(2nt-6\omega+4\Omega)\\
&-\tfrac{1169}{1536}e^3\gamma^4\sin(7nt-3\omega-4\Omega)-\tfrac{167}{512}e^3\gamma^4\sin(3nt-7\omega+4\Omega)\\
&-\tfrac{23}{1536}e\gamma^6\sin(nt-7\omega+6\Omega)-\tfrac{161}{1536}e\gamma^6\sin(7nt-\omega-6\Omega)\\
&+\tfrac{5}{384}e\gamma^6\sin(nt+5\omega-6\Omega)+\tfrac{25}{384}e\gamma^6\sin(5nt+\omega-6\Omega)\\
&-\tfrac{1659}{1280}e\,\gamma^2\sin(7nt-5\omega-2\Omega)-\tfrac{237}{256}e^5\gamma^2\sin(5nt-7\omega+2\Omega)\\
&-\tfrac{1}{64}\gamma^6\sin(4nt+2\omega-6\Omega)-\tfrac{1}{128}\gamma^6\sin(2nt+4\omega-6\Omega)\\
&+\tfrac{591}{384}e^4\gamma^2\sin(6nt-4\omega-2\Omega)+\tfrac{394}{384}e^4\gamma^2\sin(4nt-6\omega+2\Omega)
\end{aligned}\right\}, \qquad (98)
$$

$$\tfrac{1}{6}\frac{d^2 f(nt)^3}{n^2dt^2} =$$

$$\left.\begin{aligned}
&-\{e^3+\tfrac{11}{96}e^5+\tfrac{3}{32}e\gamma^4+\tfrac{119}{2304}e^7-\tfrac{3}{32}e\gamma^6-\tfrac{23}{96}e^3\gamma^4\}\sin(nt-\omega)\\
&+\{3e^4+\tfrac{115}{128}e^6+\tfrac{19}{32}e^2\gamma^4\}\sin 2(nt-\omega)\\
&+\{3e^3-\tfrac{273}{64}e^5+\tfrac{9}{32}e\gamma^4-\tfrac{2763}{1280}e^7-\tfrac{9}{32}e\gamma^6-\tfrac{327}{128}e^3\gamma^4\}\sin 3(nt-\omega)\\
&-\{6e^4-\tfrac{7}{2}e^6+\tfrac{19}{16}e^2\gamma^4\}\sin 4(nt-\omega)-\tfrac{1017}{128}e^6\sin 6(nt-\omega)\\
&+\{\tfrac{1475}{192}e^5-\tfrac{2995}{2304}e^7+\tfrac{1175}{384}e^3\gamma^4\}\sin 5(nt-\omega)+\tfrac{83153}{11520}e^7\sin 7(nt-\omega)\\
&+\{\tfrac{3}{2}e^2\gamma^2-\tfrac{3}{4}e^2\gamma^4-\tfrac{149}{192}e^4\gamma^2+\tfrac{3}{128}\gamma^6\}\sin 2(nt-\Omega)\\
&+\tfrac{27}{16}e^2\gamma^4\sin 4(nt-\Omega)-\tfrac{3}{128}\gamma^6\sin 6(nt-\Omega)-\tfrac{27}{128}e^2\gamma^4\sin(2nt-6\omega+4\Omega)\\
&-\{\tfrac{27}{64}e\gamma^4-\tfrac{27}{64}e\gamma^6-\tfrac{75}{128}e^3\gamma^4\}\sin(3nt+\omega-4\Omega)-\tfrac{243}{128}e^2\gamma^4\sin(6nt-2\omega-4\Omega)\\
&+\tfrac{3}{64}e\gamma^4-\tfrac{3}{64}e\gamma^6-\tfrac{25}{384}e^3\gamma^4\}\sin(nt+3\omega-4\Omega)\\
&+\tfrac{1}{64}e\gamma^4-\tfrac{1}{64}e\gamma^6-\tfrac{65}{384}e^3\gamma^4\}\sin(nt-5\omega+4\Omega)\\
&+\{\tfrac{25}{64}e\gamma^4-\tfrac{25}{64}e\gamma^6-\tfrac{1625}{384}e^3\gamma^4\}\sin(5nt-\omega-4\Omega)\\
&-\{\tfrac{7}{16}e^3\gamma^2-\tfrac{7}{32}e^3\gamma^4-\tfrac{209}{1536}e^5\gamma^2+\tfrac{5}{256}e\gamma^6\}\sin(nt-3\omega+2\Omega)\\
&-\{\tfrac{63}{16}e^3\gamma^2-\tfrac{63}{32}e^3\gamma^4-\tfrac{627}{512}e^5\gamma^2+\tfrac{45}{256}e\gamma^6\}\sin(3nt-\omega-2\Omega)\\
&+\{\tfrac{175}{32}e^3\gamma^2-\tfrac{175}{64}e^3\gamma^4-\tfrac{3025}{384}e^5\gamma^2+\tfrac{125}{512}e\gamma^6\}\sin(5nt-3\omega-2\Omega)\\
&+\tfrac{63}{32}e^3\gamma^2-\tfrac{63}{64}e^3\gamma^4-\tfrac{471}{128}e^5\gamma^2+\tfrac{45}{512}e\gamma^6\}\sin(3nt-5\omega+2\Omega)\\
&-\{2e^2\gamma^2-e^2\gamma^4-\tfrac{731}{96}e^4\gamma^2+\tfrac{1}{32}\gamma^6\}\sin(4nt-2\omega-2\Omega)\\
&-\{\tfrac{1}{2}e^2\gamma^2-\tfrac{1}{4}e^2\gamma^4-\tfrac{731}{384}e^4\gamma^2+\tfrac{1}{128}\gamma^6\}\sin(2nt-4\omega+2\Omega)\\
&-\tfrac{879}{96}e^4\gamma^2\sin(4nt-6\omega+2\Omega)-\tfrac{1137}{128}e^4\gamma^2\sin(6nt-4\omega-2\Omega)\\
&-\tfrac{1}{128}\gamma^6\sin(2nt+4\omega-6\Omega)+\tfrac{1}{32}\gamma^6\sin(4nt+2\omega-6\Omega)\\
&+\tfrac{833}{192}e^3\gamma^4\sin(7nt-3\omega-4\Omega)+\tfrac{51}{64}e^3\gamma^4\sin(3nt-7\omega+4\Omega)\\
&+\tfrac{8725}{1536}e^5\gamma^2\sin(5nt-7\omega+2\Omega)+\tfrac{17101}{1536}e^5\gamma^2\sin(7nt-5\omega-2\Omega)\\
&-\tfrac{75}{256}e\gamma^6\sin(5nt+\omega-6\Omega)+\tfrac{3}{256}e\gamma^6\sin(nt+5\omega-6\Omega)\\
&+\tfrac{147}{512}e\gamma^6\sin(7nt-\omega-6\Omega)+\tfrac{3}{512}e\gamma^6\sin(nt-7\omega+6\Omega)
\end{aligned}\right\} ; \quad (99)$$

$$
\left.
\begin{aligned}
\frac{1}{24}\,\frac{d^3 f(nt)^4}{n^3 dt^3} = &-\{\tfrac{1}{4}e^5 + \tfrac{67}{512}e^7 + \tfrac{11}{128}e^3\gamma^4\}\sin(nt-\omega) \\
&-\{\tfrac{8}{3}e^4 - \tfrac{37}{48}e^6 + \tfrac{1}{2}e^2\gamma^4\}\sin 2(nt-\omega) \\
&+\{\tfrac{81}{8}e^5 + \tfrac{513}{512}e^7 + \tfrac{891}{256}e^3\gamma^4\}\sin 3(nt-\omega) \\
&+\{\tfrac{16}{3}e^4 - \tfrac{59}{8}e^6 + e^2\gamma^4\}\sin 4(nt-\omega) \\
&-\{\tfrac{125}{8}e^5 - \tfrac{38375}{1536}e^7 + \tfrac{3375}{768}e^3\gamma^4\}\sin 5(nt-\omega) \\
&+\tfrac{1305}{48}e^6\sin 6(nt-\omega) - \tfrac{55909}{1536}e^7\sin 7(nt-\omega) \\
&-\{\tfrac{1}{4}e^3\gamma^2 - \tfrac{1}{8}e^3\gamma^4 + \tfrac{11}{384}e^5\gamma^2 + \tfrac{9}{768}e\gamma^6\}\sin(nt+\omega-2\Omega) \\
&+\{\tfrac{9}{2}e^3\gamma^2 - \tfrac{9}{4}e^3\gamma^4 - \tfrac{1701}{512}e^5\gamma^2 + \tfrac{27}{128}e\gamma^6\}\sin(3nt-\omega-2\Omega) \\
&+\{\tfrac{1}{6}e^3\gamma^2 - \tfrac{1}{12}e^3\gamma^4 - \tfrac{189}{1536}e^5\gamma^2 + \tfrac{1}{128}e\gamma^6\}\sin(nt-3\omega+2\Omega) \\
&-\{\tfrac{9}{8}e^3\gamma^2 - \tfrac{9}{16}e^3\gamma^4 - \tfrac{1539}{256}e^5\gamma^2 + \tfrac{27}{512}e\gamma^6\}\sin(3nt-5\omega+2\Omega) \\
&-\{\tfrac{125}{24}e^3\gamma^2 - \tfrac{125}{48}e^3\gamma^4 - \tfrac{21375}{768}e^5\gamma^2 + \tfrac{125}{512}e\gamma^6\}\sin(5nt-3\omega-2\Omega) \\
&+\tfrac{3}{8}e^2\gamma^4\sin(2nt+2\omega-4\Omega) - \tfrac{27}{256}e\gamma^6\sin(3nt+3\omega-6\Omega) \\
&-\tfrac{343}{1536}e\gamma^6\sin(7nt-\omega-6\Omega) - \tfrac{1}{1536}e\gamma^6\sin(nt-7\omega+6\Omega) \\
&-\tfrac{16375}{1536}e^5\gamma^2\sin(5nt-7\omega+2\Omega) - \tfrac{44933}{1536}e^5\gamma^2\sin(7nt-5\omega-2\Omega) \\
&-\tfrac{369}{256}e^3\gamma^4\sin(3nt+\omega-4\Omega) - \tfrac{41}{768}e^3\gamma^4\sin(nt+3\omega-4\Omega) \\
&-\tfrac{369}{512}e^3\gamma^4\sin(3nt-7\omega+4\Omega) - \tfrac{14063}{1536}e^3\gamma^4\sin(7nt-3\omega-4\Omega) \\
&+\tfrac{1}{16}e^2\gamma^4\sin(2nt-6\omega+4\Omega) + \tfrac{27}{16}e^2\gamma^4\sin(6nt-2\omega-4\Omega) \\
&-\tfrac{17}{8}e^4\gamma^2\sin(2nt-4\omega+2\Omega) - 17e^4\gamma^2\sin(4nt-2\omega-2\Omega) \\
&+\tfrac{17}{8}e^4\gamma^2\sin(4nt-6\omega+2\Omega) + \tfrac{153}{8}e^4\gamma^2\sin(6nt-4\omega-2\Omega) \\
&+\tfrac{41}{512}e^3\gamma^4\sin(nt-5\omega+4\Omega) + \tfrac{5125}{512}e^3\gamma^4\sin(5nt-\omega-4\Omega) \\
&-2e^2\gamma^4\sin 4(nt-\Omega) + \tfrac{17}{12}e^4\gamma^2\sin 2(nt-\Omega) \\
&+\tfrac{1}{384}e\gamma^6\sin(nt+5\omega-6\Omega) + \tfrac{125}{384}e\gamma^6\sin(5nt+\omega-6\Omega)
\end{aligned}
\right\}, \qquad (100)
$$

$$\left.\begin{aligned}
\tfrac{1}{120}\frac{d^4 f(nt)^5}{n^4dt^4} &= \{\tfrac{1}{6}e^5+\tfrac{55}{1152}e^7+\tfrac{5}{96}e^3\gamma^4\}\sin(nt-\omega)-\tfrac{5}{2}e^6\sin 2(nt-\omega)\\
&-\{\tfrac{27}{4}e^5-\tfrac{1107}{128}e^7+\tfrac{135}{64}e^3\gamma^4\}\sin 3(nt-\omega)+32e^6\sin 4(nt-\omega)\\
&+\{\tfrac{125}{12}e^5-\tfrac{90625}{1152}e^7+\tfrac{625}{192}e^3\gamma^4\}\sin 5(nt-\omega)-\tfrac{81}{2}e^6\sin 6(nt-\omega)\\
&+\tfrac{103243}{1152}e^7\sin 7(nt-\omega)-\tfrac{5}{3}e^4\gamma^2\sin 2(nt-\Omega)\\
&+\tfrac{5}{6}e^4\gamma^2\sin(2nt-4\omega+2\Omega)+\tfrac{40}{3}e^4\gamma^2\sin(4nt-2\omega-2\Omega)\\
&-\tfrac{8}{3}e^4\gamma^2\sin(4nt-6\omega+2\Omega)-\tfrac{27}{2}e^4\gamma^2\sin(6nt-4\omega-2\Omega)\\
&+\tfrac{3125}{192}e^5\gamma^2\sin(5nt-7\omega+2\Omega)+\tfrac{12005}{192}e^5\gamma^2\sin(7nt-5\omega-2\Omega)\\
&+\tfrac{25}{192}e^5\gamma^2\sin(nt-3\omega+2\Omega)+\tfrac{2025}{192}e^5\gamma^2\sin(3nt-\omega-2\Omega)\\
&-\tfrac{405}{48}e^5\gamma^2\sin(3nt-5\omega+2\Omega)-\tfrac{3125}{48}e^5\gamma^2\sin(5nt-3\omega-2\Omega)\\
&+\tfrac{135}{64}e^3\gamma^4\sin(3nt+\omega-4\Omega)-\tfrac{5}{192}e^3\gamma^4\sin(nt+3\omega-4\Omega)\\
&-\tfrac{5}{384}e^3\gamma^4\sin(nt-5\omega+4\Omega)-\tfrac{3125}{384}e^3\gamma^4\sin(5nt-\omega-4\Omega)\\
&+\tfrac{27}{128}e^3\gamma^4\sin(3nt-7\omega+4\Omega)+\tfrac{2401}{384}e^3\gamma^4\sin(7nt-3\omega-4\Omega)
\end{aligned}\right\};\tag{101}$$

$$\left.\begin{aligned}
\tfrac{1}{720}\frac{d^5 f(nt)^6}{n^5dt^5} &= +\tfrac{1}{32}e^7\sin(nt-\omega)+\tfrac{4}{3}e^6\sin 2(nt-\omega)\\
&-\tfrac{2187}{160}e^7\sin 3(nt-\omega)-\tfrac{256}{15}e^6\sin 4(nt-\omega)\\
&+\tfrac{3125}{32}e^7\sin 5(nt-\omega)+\tfrac{108}{5}e^6\sin 6(nt-\omega)\\
&+\tfrac{1}{24}e^5\gamma^2\sin(nt+\omega-2\Omega)-\tfrac{243}{32}e^5\gamma^2\sin(3nt-\omega-2\Omega)\\
&-\tfrac{1}{32}e^5\gamma^2\sin(nt-3\omega+2\Omega)+\tfrac{625}{16}e^5\gamma^2\sin(5nt-3\omega-2\Omega)\\
&+\tfrac{243}{80}e^5\gamma^2\sin(3nt-5\omega+2\Omega)-\tfrac{16807}{480}e^5\gamma^2\sin(7nt-5\omega-2\Omega)\\
&-\tfrac{625}{96}e^5\gamma^2\sin(5nt-7\omega+2\Omega)-\tfrac{16807}{160}e^7\sin 7(nt-\omega)
\end{aligned}\right\};\tag{102}$$

$$\left.\begin{aligned}
\tfrac{1}{5040}\frac{d^6 f(nt)^7}{n^6dt^6} &= -\tfrac{1}{72}e^7\sin(nt-\omega)+\tfrac{243}{40}e^7\sin 3(nt-\omega)\\
&-\tfrac{3125}{72}e^7\sin 5(nt-\omega)+\tfrac{16807}{360}e^7\sin 7(nt-\omega)
\end{aligned}\right\}.\tag{103}$$

If we now substitute equation (91) and equations (98–103) in equation (90), we shall obtain the following value of the true longitude v of the moon in terms of the mean longitude nt.

$$
\left.
\begin{aligned}
v = nt & \\
& + \{2e - \tfrac{1}{4}e^3 + \tfrac{5}{96}e^5 - \tfrac{1}{8}e\gamma^4 + \tfrac{1}{8}e\gamma^6 + \tfrac{107}{4608}e^7 + \tfrac{5}{48}e^3\gamma^4\} \sin(nt - \omega) \\
& + \{\tfrac{5}{4}e^2 - \tfrac{11}{24}e^4 + \tfrac{1}{16}\gamma^4 + \tfrac{17}{192}e^6 - \tfrac{1}{4}e^2\gamma^4 - \tfrac{1}{16}\gamma^6\} \sin 2(nt - \omega) \\
& + \{\tfrac{13}{12}e^3 - \tfrac{43}{64}e^5 + \tfrac{1}{8}e\gamma^4 - \tfrac{1}{8}e\gamma^6 + \tfrac{95}{512}e^7 - \tfrac{27}{64}e^3\gamma^4\} \sin 3(nt - \omega) \\
& + \{\tfrac{103}{96}e^4 - \tfrac{451}{480}e^6 + \tfrac{13}{64}e^2\gamma^4\} \sin 4(nt - \omega) + \tfrac{1223}{960}e^6 \sin 6(nt - \omega) \\
& + \{\tfrac{1097}{960}e^5 - \tfrac{5957}{4608}e^7 + \tfrac{493}{384}e^3\gamma^4\} \sin 5(nt - \omega) + \tfrac{47273}{32256}e^7 \sin 7(nt - \omega) \\
& - \{\tfrac{1}{4}\gamma^2 - \tfrac{1}{8}\gamma^4 - e^2\gamma^2 + \tfrac{1}{2}e^2\gamma^4 + \tfrac{55}{64}e^4\gamma^2 + \tfrac{1}{16}\gamma^6\} \sin 2(nt - \Omega) \\
& + \{\tfrac{1}{32}\gamma^4 - \tfrac{1}{32}\gamma^6 - \tfrac{45}{64}e^2\gamma^4\} \sin 4(nt - \Omega) - \tfrac{1}{192}\gamma^6 \sin 6(nt - \Omega) \\
& + \{\tfrac{1}{4}\gamma^2 - \tfrac{1}{8}\gamma^4 - \tfrac{3}{16}e^2\gamma^2 - \tfrac{1}{32}e^4\gamma^2 + \tfrac{5}{64}\gamma^6 + \tfrac{3}{32}e^2\gamma^4\} \sin 2(\omega - \Omega) \\
& + \{\tfrac{1}{32}\gamma^4 - \tfrac{1}{32}\gamma^6 - \tfrac{3}{64}e^2\gamma^4\} \sin 4(\omega - \Omega + \tfrac{1}{192}\gamma^6 \sin 6(\omega - \Omega) \\
& + \{\tfrac{1}{2}e\gamma^2 - \tfrac{1}{4}e\gamma^4 - \tfrac{7}{16}e^3\gamma^2 + \tfrac{7}{32}e^3\gamma^4 + \tfrac{1}{8}e\gamma^6 - \tfrac{19}{384}e^5\gamma^2\} \sin(nt + \omega - 2\Omega) \\
& - \{\tfrac{1}{2}e\gamma^2 - \tfrac{1}{4}e\gamma^4 - \tfrac{27}{16}e^3\gamma^2 + \tfrac{27}{32}e^3\gamma^4 + \tfrac{1}{16}e\gamma^6 - \tfrac{117}{128}e^5\gamma^2\} \sin(3nt - \omega - 2\Omega) \\
& - \{\tfrac{13}{16}e^2\gamma^2 - \tfrac{13}{32}e^2\gamma^4 - \tfrac{259}{96}e^4\gamma^2 + \tfrac{1}{64}\gamma^6\} \sin(4nt - 2\omega - 2\Omega) \\
& - \{\tfrac{1}{16}\gamma^4 - \tfrac{1}{16}\gamma^6 - \tfrac{27}{64}e^2\gamma^4\} \sin(2nt + 2\omega - 4\Omega) \\
& + \{\tfrac{1}{48}e^3\gamma^2 - \tfrac{1}{96}e^3\gamma^4 + \tfrac{77}{768}e^5\gamma^2\} \sin(nt - 3\omega + 2\Omega) \\
& - \{\tfrac{1}{4}e\gamma^4 - \tfrac{1}{4}e\gamma^6 - \tfrac{9}{8}e^3\gamma^4\} \sin(3nt + \omega - 4\Omega) \\
& + \{\tfrac{1}{8}e\gamma^4 - \tfrac{1}{8}e\gamma^6 - \tfrac{19}{96}e^3\gamma^4\} \sin(nt + 3\omega - 4\Omega) \\
& + \{\tfrac{1}{8}e\gamma^4 - \tfrac{1}{8}e\gamma^6 - \tfrac{157}{96}e^3\gamma^4\} \sin(5nt - \omega - 4\Omega) \\
& - \{\tfrac{59}{48}e^3\gamma^2 - \tfrac{59}{96}e^3\gamma^4 + \tfrac{1}{16}e\gamma^6 + \tfrac{5779}{768}e^5\gamma^2\} \sin(5nt - 3\omega - 2\Omega) \\
& + \{\tfrac{1}{96}e^4\gamma^2 \sin(2nt - 4\omega + 2\Omega) + \tfrac{21}{64}e^2\gamma^4 \sin(6nt - 2\omega - 4\Omega) \\
& - \tfrac{115}{64}e^4\gamma^2 \sin(6nt - 4\omega - 2\Omega) + \tfrac{1}{64}\gamma^6 \sin(4nt + 2\omega - 6\Omega) \\
& - \tfrac{1}{64}\gamma^6 \sin(2nt + 4\omega - 6\Omega) - \tfrac{3231}{1280}e^5\gamma^2 \sin(3nt - 5\omega + 2\Omega) \\
& - \tfrac{11}{48}e\gamma^6 \sin(5nt + \omega - 6\Omega) + \tfrac{17}{48}e\gamma^6 \sin(nt + 5\omega - 6\Omega) \\
& - \tfrac{1}{32}e\gamma^6 \sin(7nt - \omega - 6\Omega) - \tfrac{3}{32}e\gamma^6 \sin(3nt + 3\omega - 6\Omega) \\
& + \tfrac{137}{192}e^3\gamma^4 \sin(7nt - 3\omega - 4\Omega) + \tfrac{31267}{3840}e^5\gamma^2 \sin(7nt - 5\omega - 2\Omega) \\
& + \tfrac{125}{32}e^5\gamma^2 \sin(5nt - 7\omega + 2\Omega) + \tfrac{1}{192}e^3\gamma^4 \sin(nt - 5\omega + 4\Omega)
\end{aligned}
\right\} \quad (104)
$$

6. Having found the value of v in terms of nt, if we now substitute it in equations (73) and (39), we shall also obtain the values of r and θ in terms of nt. For this purpose we shall first develop equation (73) in a series, carrying the approximation to terms of the fifth order, and we shall find

$$\left.\begin{aligned}\frac{a(1-e^2)}{r} &= 1 - e\{1-\tfrac{1}{64}\gamma^4\}\cos(v-\omega) - \tfrac{1}{4}e\gamma^2\{1-\tfrac{1}{2}\gamma^2\}\cos(v+\omega-2\Omega) \\ &+ \tfrac{1}{8}e\gamma^2\{1-\tfrac{1}{2}\gamma^2\}\cos\{v-3\omega+2\Omega) + \tfrac{1}{8}e\gamma^2\{1-\tfrac{1}{2}\gamma^2\}\cos(3v-\omega-2\Omega) \\ &+ \tfrac{3}{128}e\gamma^4\cos(v-5\omega+4\Omega) + \tfrac{3}{128}e\gamma^4\cos(5v-\omega-4\Omega) \\ &- \tfrac{3}{128}e\gamma^4\cos(3v+\omega-4\Omega) - \tfrac{3}{128}e\gamma^4\cos(v+3\omega-4\Omega) + \tfrac{1}{64}e\gamma^4\cos 3(v-\omega)\end{aligned}\right\}. \qquad (105)$$

In order to substitute the value of v in this equation, we shall observe that if we put all the terms of the second member of equation (104), except the first, equal to β, we shall have

$$v = nt + \beta. \qquad (106)$$

This gives
$$mv + a = (mnt + a) + m\beta, \qquad (107)$$

m being any positive whole number, and a any angle whatever.

Equation (107) gives

$$\cos(mv + a) = \cos(mnt + a)\cos m\beta - \sin(mnt + a)\sin m\beta. \qquad (108)$$

Since β is a small quantity, we may develop $\sin m\beta$ and $\cos m\beta$ in a series, and we shall have, with sufficient accuracy,

$$\sin m\beta = m\beta - \tfrac{1}{6}m^3\beta^3, \qquad \cos m\beta = 1 - \tfrac{1}{2}m^2\beta^2 + \tfrac{1}{24}m^4\beta^4. \qquad (109)$$

We shall then find, as far as terms of the fourth order,

$$\left.\begin{aligned}\beta^2 &= 2e^2 + \tfrac{9}{32}e^4 + \tfrac{1}{16}\gamma^4 + \tfrac{5}{2}e^3\cos(nt-\omega) \\ &- \{2e^2 - \tfrac{8}{3}e^4 + \tfrac{1}{16}\gamma^4\}\cos 2(nt-\omega) - \tfrac{5}{2}e^3\cos 3(nt-\omega) \\ &- \tfrac{283}{96}e^4\cos 4(nt-\omega) - \tfrac{37}{16}e^2\gamma^2\cos 2(nt-\Omega) + \tfrac{11}{16}e^2\gamma^2\cos 2(\omega-\Omega) \\ &- \tfrac{1}{32}\gamma^4\cos 4(nt-\Omega) - \tfrac{1}{32}\gamma^4\cos 4(\omega-\Omega) + \tfrac{1}{16}\gamma^4\cos(2nt+2\omega-4\Omega) \\ &- e\gamma^2\cos(nt+\omega-2\Omega) + \tfrac{1}{2}e\gamma^2\cos(3nt-\omega-2\Omega) + \tfrac{1}{2}e\gamma^2\cos(nt-3\omega+2\Omega) \\ &+ \tfrac{21}{16}e^2\gamma^2\cos(4nt-2\omega-2\Omega) + \tfrac{5}{16}e^2\gamma^2\cos(2nt-4\omega+2\Omega)\end{aligned}\right\}. \qquad (110)$$

$$\left.\begin{aligned}\beta^3 = {} & 6e^3 \sin(nt-\omega) + \tfrac{15}{2}e^4 \sin 2(nt-\omega) - 2e^3 \sin 3(nt-\omega) \\ & - \tfrac{15}{4}e^4 \sin 4(nt-\omega) - \tfrac{9}{4}e^2\gamma^2 \sin 2(nt-\Omega) + \tfrac{9}{4}e^2\gamma^2 \sin 2(\omega-\Omega) \\ & + \tfrac{3}{4}e^2\gamma^2 \sin(4nt-2\omega-2\Omega) + \tfrac{3}{4}e^2\gamma^2 \sin(2nt-4\omega+2\Omega)\end{aligned}\right\}, \tag{111}$$

$$\beta^4 = 6e^4 - 8e^4 \cos 2(nt-\omega) + 2e^4 \cos 4(nt-\omega) \tag{112}$$

Therefore equation (108) will become

$$\left.\begin{aligned}\cos(mv+a) = {} & \cos(mnt+a) \times \\ & \{1 - m^2(e^2 + \tfrac{9}{64}e^4 + \tfrac{1}{32}\gamma^4) + \tfrac{1}{4}m^4e^4 - \tfrac{5}{4}m^2e^3 \cos(nt-\omega) \\ & + (m^2e^2 - \tfrac{4}{3}m^2e^4 + \tfrac{1}{32}m^2\gamma^4 - \tfrac{1}{3}m^4e^4) \cos 2(nt-\omega) \\ & + \tfrac{5}{4}m^2e^3 \cos 3(nt-\omega) + (\tfrac{283}{192}m^2e^4 + \tfrac{1}{12}m^4e^4) \cos 4(nt-\omega) \\ & + \tfrac{37}{32}m^2e^2\gamma^2 \cos 2(nt-\Omega) - \tfrac{11}{32}m^2e^2\gamma^2 \cos 2(\omega-\Omega) \\ & + \tfrac{1}{64}m^2\gamma^4 \cos 4(nt-\Omega) + \tfrac{1}{64}m^2\gamma^4 \cos 4(\omega-\Omega) \\ & - \tfrac{1}{32}m^2\gamma^4 \cos(2nt+2\omega-4\Omega) + \tfrac{1}{2}e\gamma^2m^2 \cos(nt+\omega-2\Omega) \\ & - \tfrac{1}{4}m^2e\gamma^2 \cos(3nt-\omega-2\Omega) - \tfrac{1}{4}m^2e\gamma^2 \cos(nt-3\omega+2\Omega) \\ & - \tfrac{21}{32}m^2e^2\gamma^2 \cos(4nt-2\omega-2\Omega) - \tfrac{5}{32}m^2e^2\gamma^2 \cos(2nt-4\omega+2\Omega)\} \\ & - \sin(mnt+a) \times \\ & \{(2me - \tfrac{1}{4}me^3 - m^3e^3) \sin(nt-\omega) + (\tfrac{5}{4}me^2 - \tfrac{11}{24}me^4 + \tfrac{1}{16}m\gamma^4 - \tfrac{5}{4}m^3e^4) \sin 2(nt-\omega) \\ & + (\tfrac{13}{12}me^3 + \tfrac{1}{3}m^3e^3) \sin 3(nt-\omega) + (\tfrac{103}{96}me^4 + \tfrac{5}{8}m^3e^4) \sin 4(nt-\omega) \\ & - (\tfrac{1}{4}m\gamma^2 - \tfrac{1}{8}m\gamma^4 - me^2\gamma^2 - \tfrac{3}{8}m^3e^2\gamma^2) \sin 2(nt-\Omega) - \tfrac{1}{2}me\gamma^2 \sin(3nt-\omega-2\Omega) \\ & + (\tfrac{1}{4}m\gamma^2 - \tfrac{1}{8}m\gamma^4 - \tfrac{3}{16}me^2\gamma^2 - \tfrac{3}{8}m^3e^2\gamma^2) \sin 2(\omega-\Omega) + \tfrac{1}{2}me\gamma^2 \sin(nt+\omega-2\Omega) \\ & + \tfrac{1}{32}m\gamma^4 \sin 4(nt-\Omega) + \tfrac{1}{32}m\gamma^4 \sin 4(\omega-\Omega) - \tfrac{1}{16}m\gamma^4 \sin(2nt+2\omega-4\Omega) \\ & - (\tfrac{13}{16}me^2\gamma^2 + \tfrac{1}{8}m^3e^2\gamma^2) \sin(4nt-2\omega-2\Omega) - \tfrac{1}{8}m^3e^2\gamma^2 \sin(2nt-4\omega+2\Omega)\}\end{aligned}\right\} \tag{113}$$

If in this equation we put in succession $m=1$, $a=-\omega$, $m=1$, $a=\omega-2\Omega$, $m=1$, $a=-3\omega+2\Omega$, and $m=3$, $a=-\omega-2\Omega$, we shall get the following equations

$$\left.\begin{aligned}
\cos(v-\omega) = & -e + \{1 - \tfrac{9}{8}e^2 + \tfrac{25}{192}e^4 - \tfrac{3}{64}\gamma^4\}\cos(nt-\omega) \\
& + \{e - \tfrac{4}{3}e^3\}\cos 2(nt-\omega) + \{\tfrac{9}{8}e^2 - \tfrac{225}{128}e^4 + \tfrac{3}{64}\gamma^4\}\cos 3(nt-\omega) \\
& + \tfrac{4}{3}e^3\cos 4(nt-\omega) + \tfrac{625}{384}e^4\cos 5(nt-\omega) \\
& - \{\tfrac{1}{8}\gamma^2 - \tfrac{1}{16}\gamma^4 - \tfrac{45}{32}e^2\gamma^2\}\cos(3nt-\omega-2\Omega) \\
& + \{\tfrac{1}{4}\gamma^2 - \tfrac{1}{8}\gamma^4 - \tfrac{9}{16}e^2\gamma^2\}\cos(nt+\omega-2\Omega) \\
& - \{\tfrac{1}{8}\gamma^2 - \tfrac{1}{16}\gamma^4 - \tfrac{3}{32}e^2\gamma^2\}\cos(nt-3\omega+2\Omega) \\
& + \tfrac{3}{128}\gamma^4\cos(5nt-\omega-4\Omega) - \tfrac{7}{128}\gamma^4\cos(3nt+\omega-4\Omega) \\
& + \tfrac{5}{128}\gamma^4\cos(nt+3\omega-4\Omega) - \tfrac{1}{128}\gamma^4\cos(nt-5\omega+4\Omega) \\
& + \tfrac{5}{8}e\gamma^2\cos 2(nt-\Omega) - \tfrac{3}{8}e\gamma^2\cos(4nt-2\omega-2\Omega) \\
& - \tfrac{1}{8}e\gamma^2\cos(2nt-4\omega+2\Omega) - \tfrac{1}{8}e\gamma^2\cos 2(\omega-\Omega) \\
& - \tfrac{51}{64}e^2\gamma^2\cos(5nt-3\omega-2\Omega) - \tfrac{9}{64}e^2\gamma^2\cos(3nt-5\omega+2\Omega)
\end{aligned}\right\} ; \quad (114)$$

$$\left.\begin{aligned}
\cos(v+\omega-2\Omega) = & (1-e^2)\cos(nt+\omega-2\Omega) + \tfrac{3}{8}e^2\cos(3nt-\omega-2\Omega) \\
& - \tfrac{1}{8}e^2\cos(nt-3\omega+2\Omega) - \tfrac{1}{8}\gamma^2\cos(3nt+\omega-4\Omega) \\
& + \tfrac{1}{8}\gamma^2\cos(nt+3\omega-4\Omega) + e\cos 2(nt-\Omega) - e\cos 2(\omega-\Omega)
\end{aligned}\right\} ; \quad (115)$$

$$\left.\begin{aligned}
\cos(v-3\omega+2\Omega) = & (1-e^2)\cos(nt-3\omega+2\Omega) - \tfrac{1}{8}e^2\cos(nt+\omega-2\Omega) \\
& + \tfrac{9}{8}e^2\cos(3nt-5\omega+2\Omega) + \tfrac{1}{8}\gamma^2\cos(nt+3\omega-4\Omega) \\
& - \tfrac{1}{8}\gamma^2\cos(nt-5\omega+4\Omega) + e\cos(2nt-4\omega+2\Omega) \\
& - e\cos 2(\omega-\Omega) + \tfrac{1}{8}\gamma^2\cos(nt-\omega) - \tfrac{1}{8}\gamma^2\cos 3(nt-\omega)
\end{aligned}\right\} ; \quad (116)$$

$$\left.\begin{aligned}
\cos(3v-\omega-2\Omega) = & (1-9e^2)\cos(3nt-\omega-2\Omega) + \tfrac{51}{8}e^2\cos(5nt-3\omega-2\Omega) \\
& + \tfrac{21}{8}e^2\cos(nt+\omega-2\Omega) + \tfrac{3}{8}\gamma^2\cos(3nt+\omega-4\Omega) \\
& - \tfrac{3}{8}\gamma^2\cos(5nt-\omega-4\Omega) + 3e\cos(4nt-2\omega-2\Omega) \\
& - 3e\cos 2(nt-\Omega) + \tfrac{3}{8}\gamma^2\cos(nt-\omega) - \tfrac{3}{8}\gamma^2\cos 3(nt-\omega)
\end{aligned}\right\} . \quad (117)$$

If we now substitute these values in equation (105), and at the same time

change v into nt in the terms of the fifth order in that equation, it will become

$$
\left.\begin{aligned}
\frac{a(1-e^2)}{r} = 1 - e^2 + e\left\{1 - \tfrac{9}{8}e^2 + \tfrac{25}{192}e^4\right\}\cos(nt-\omega) \\
+ e^2\left\{1 - \tfrac{4}{3}e^2\right\}\cos 2(nt-\omega) + \left\{\tfrac{9}{8}e^3 - \tfrac{225}{128}e^5\right\}\cos 3(nt-\omega) \\
+ \tfrac{4}{3}e^4\cos 4(nt-\omega) + \tfrac{625}{384}e^5\cos 5(nt-\omega)
\end{aligned}\right\} \qquad (118)
$$

In the substitution in equation (105), the coefficients of all the terms containing Ω become identically equal to nothing, and the quantity γ disappears from the coefficients of all the remaining terms, whence it follows that the radius vector of the orbit is entirely independent of the position of the orbit We shall therefore have

$$
\left.\begin{aligned}
\frac{a}{r} = 1 + e\left\{1 - \tfrac{1}{8}e^2 + \tfrac{1}{192}e^4\right\}\cos(nt-\omega) + e^2\left\{1 - \tfrac{1}{3}e^2\right\}\cos 2(nt-\omega) \\
+ \left\{\tfrac{9}{8}e^3 - \tfrac{81}{128}e^5\right\}\cos 3(nt-\omega) + \tfrac{4}{3}e^4\cos 4(nt-\omega) + \tfrac{625}{384}e^5\cos 5(nt-\omega)
\end{aligned}\right\} \qquad (119)
$$

In order to find the tangent of the latitude it is only necessary to find $\sin(v-\Omega)$, and multiply it by γ To find $\sin(v-\Omega)$ we shall observe that if we change $\cos(mnt+a)$ into $\sin(mnt+a)$, and $-\sin(mnt+a)$ into $+\cos(mnt+a)$, $\cos(mv+a)$ will change $\sin(mv+a)$ If we then make these changes in equation (113) and put $m=1$, $a=-\Omega$, we shall get, after multiplying by γ, the following value of $\gamma\sin(v-\Omega)$, or $\tan\theta$,

$$
\left.\begin{aligned}
\tan\theta = \gamma\left\{1 - e^2 - \tfrac{1}{8}\gamma^2 + \tfrac{7}{64}e^4 + \tfrac{1}{32}\gamma^4 + \tfrac{7}{64}e^2\gamma^2\right\}\sin(nt-\Omega) \\
- \left\{\tfrac{1}{8}\gamma^3 - \tfrac{9}{128}\gamma^5 - \tfrac{81}{64}e^2\gamma^3\right\}\sin 3(nt-\Omega) + \tfrac{3}{128}\gamma^5\sin 5(nt-\Omega) \\
+ e\gamma\left\{1 - \tfrac{5}{4}e^2 - \tfrac{1}{8}\gamma^2\right\}\sin(2nt-\omega-\Omega) - e\gamma\sin(\omega-\Omega) \\
+ \left\{\tfrac{9}{8}e^2\gamma - \tfrac{27}{16}e^4\gamma + \tfrac{3}{64}\gamma^5 - \tfrac{9}{64}e^2\gamma^3\right\}\sin(3nt-2\omega-\Omega) \\
+ \left\{\tfrac{1}{8}e^2\gamma - \tfrac{1}{8}\gamma^3 - \tfrac{1}{48}e^4\gamma + \tfrac{5}{64}\gamma^5 + \tfrac{7}{64}e^2\gamma^3\right\}\sin(nt-2\omega+\Omega) \\
+ \left\{\tfrac{1}{8}\gamma^3 - \tfrac{5}{64}\gamma^5 - \tfrac{29}{64}e^2\gamma^3\right\}\sin(nt+2\omega-3\Omega) - \tfrac{1}{8}e\gamma^3\sin 3(\omega-\Omega) \\
+ \tfrac{4}{3}e^3\gamma\sin(4nt-3\omega-\Omega) + \tfrac{4}{3}e^3\gamma\sin(4nt-3\omega-\Omega) \\
+ \left\{\tfrac{1}{12}e^3\gamma - \tfrac{1}{8}e\gamma^3\right\}\sin(2nt-3\omega+\Omega) + \tfrac{625}{384}e^4\gamma\sin(5nt-4\omega-\Omega) \\
+ \left\{\tfrac{9}{128}e^4\gamma - \tfrac{9}{64}e^2\gamma^3\right\}\sin(3nt-4\omega+\Omega) + \tfrac{3}{128}\gamma^5\sin(nt+4\omega-5\Omega) \\
+ \left\{\tfrac{1}{64}e^2\gamma^3 - \tfrac{1}{128}\gamma^5\right\}\sin(nt-4\omega+3\Omega) - \tfrac{3}{8}e\gamma^3\sin(4nt-\omega-3\Omega) \\
- \tfrac{3}{64}\gamma^5\sin(3nt+2\omega-5\Omega) + \tfrac{1}{2}e\gamma^3\sin(2nt+\omega-3\Omega) - \tfrac{51}{64}e^2\gamma^3\sin(5nt-2\omega-3\Omega)
\end{aligned}\right\} \qquad (120)
$$

This equation will give

$$\left.\begin{aligned}\tfrac{1}{3}\tan^3\theta = {} & \tfrac{1}{4}\gamma^3\{1-e^2-\tfrac{1}{4}\gamma^2\}\sin(nt-\Omega) - \tfrac{1}{12}\gamma^3\{1-9e^2+\tfrac{3}{8}\gamma^2\}\sin 3(nt-\Omega) \\ & + \tfrac{1}{32}\gamma^5\sin 5(nt-\Omega) - \tfrac{1}{4}e\gamma^3\sin(\omega-\Omega) + \tfrac{1}{4}e\gamma^3\sin(2nt+\omega-3\Omega) \\ & + \tfrac{1}{4}e\gamma^3\sin(2nt-\omega-\Omega) - \tfrac{1}{4}e\gamma^3\sin(4nt-\omega-3\Omega) - \tfrac{1}{32}\gamma^5\sin(3nt+2\omega-5\Omega) \\ & + \tfrac{1}{32}\gamma^3\{\gamma^2+9e^2\}\sin(3nt-2\omega-\Omega) - \tfrac{17}{32}e^2\gamma^3\sin(5nt-2\omega-3\Omega) \\ & + \tfrac{1}{32}\gamma^3\{e^2-\gamma^2\}\sin(nt-2\omega+\Omega) + \tfrac{1}{32}\gamma^3\{\gamma^2-7e^2\}\sin(nt+2\omega-3\Omega)\end{aligned}\right\} . \quad (121)$$

$$\tfrac{1}{5}\tan^5\theta = \tfrac{1}{8}\gamma^5\sin(nt-\Omega) - \tfrac{1}{16}\gamma^5\sin 3(nt-\Omega) + \tfrac{1}{80}\gamma^5\sin 5(nt-\Omega). \quad (122)$$

Now we have $\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta -$, etc. (123)

If we now substitute these values of $\tan\theta$ and its powers in equation (123) we shall get the following expression for the latitude of the moon, which is correct to terms of the fifth order:

$$\left.\begin{aligned}\theta = {} & \gamma\{1-e^2-\tfrac{3}{8}\gamma^2+\tfrac{7}{64}e^4+\tfrac{7}{32}\gamma^4+\tfrac{23}{64}e^2\gamma^2\}\sin(nt-\Omega) \\ & - \{\tfrac{1}{24}\gamma^3-\tfrac{5}{128}\gamma^5-\tfrac{33}{64}e^2\gamma^3\}\sin 3(nt-\Omega) + \tfrac{3}{640}\gamma^5\sin 5(nt-\Omega) \\ & + e\gamma\{1-\tfrac{5}{4}e^2-\tfrac{3}{8}\gamma^2\}\sin(2nt-\omega-\Omega) - e\gamma\{1-\tfrac{1}{4}\gamma^2\}\sin(\omega-\Omega) \\ & + \{\tfrac{9}{8}e^2\gamma-\tfrac{27}{16}e^4\gamma+\tfrac{1}{64}\gamma^5-\tfrac{27}{64}e^2\gamma^3\}\sin(3nt-2\omega-\Omega) - \tfrac{1}{8}e\gamma^3\sin 3(\omega-\Omega) \\ & + \{\tfrac{1}{8}e^2\gamma-\tfrac{1}{8}\gamma^3-\tfrac{1}{48}e^4\gamma+\tfrac{7}{64}\gamma^5+\tfrac{5}{64}e^2\gamma^3\}\sin(nt-2\omega+\Omega) \\ & + \{\tfrac{1}{8}\gamma^3-\tfrac{7}{64}\gamma^5-\tfrac{15}{64}e^2\gamma^3\}\sin(nt+2\omega-3\Omega) + \tfrac{4}{3}e^3\gamma\sin(4nt-3\omega-\Omega) \\ & - \tfrac{1}{8}e\gamma^3\sin(4nt-\omega-3\Omega) + \tfrac{1}{4}e\gamma^3\sin(2nt+\omega-3\Omega) \\ & + \{\tfrac{1}{12}e^3\gamma-\tfrac{1}{8}e\gamma^3\}\sin(2nt-3\omega+\Omega) + \tfrac{625}{384}e^4\gamma\sin(5nt-4\omega-\Omega) \\ & + \{\tfrac{9}{128}e^4\gamma-\tfrac{9}{64}e^2\gamma^3\}\sin(3nt-4\omega+\Omega) + \tfrac{3}{128}\gamma^5\sin(nt+4\omega-5\Omega) \\ & + \{\tfrac{1}{64}e^2\gamma^3-\tfrac{1}{128}\gamma^5\}\sin(nt-4\omega+3\Omega) - \tfrac{1}{64}\gamma^5\sin(3nt+2\omega-5\Omega) \\ & - \tfrac{17}{64}e^2\gamma^3\sin(5nt-2\omega-3\Omega)\end{aligned}\right\} . \quad (124)$$

7. We have thus found the values of the three co-ordinates r, v and θ in terms of the time. We may however find θ directly from the differential equation (28), which will serve as a verification of all the developments which have thus far been made in the determination of t, v, r and θ. For this purpose we shall sub-

stitute the values of c, c' and c'' in equation (28), by which means it will become

$$\frac{d\theta}{dt}=\frac{\sqrt{a\mu(1-e^2)}}{\sqrt{1+\gamma^2}}\;\frac{\gamma}{r^2}\cos(v-\Omega) \tag{125}$$

Now equation (119) will give

$$\left.\begin{aligned}\frac{a^2}{r^2}=1+\tfrac{1}{2}e^2+\tfrac{3}{8}e^4+\{2e+\tfrac{3}{4}e^3+\tfrac{65}{96}e^5\}\cos(nt-\omega)\\+\{\tfrac{5}{2}e^2+\tfrac{1}{3}e^4\}\cos 2(nt-\omega)+\tfrac{103}{24}e^4\cos 4(nt-\omega)\\+\{\tfrac{13}{4}e^3-\tfrac{25}{64}e^5\}\cos 3(nt-\omega)+\tfrac{1097}{192}e^5\cos 5(nt-\omega)\end{aligned}\right\}, \tag{126}$$

and equation (113) gives, by putting $m=1$, and $a=-\Omega$,

$$\left.\begin{aligned}\cos(v-\Omega)=\{1-e^2+\tfrac{1}{8}\gamma^2+\tfrac{7}{64}e^4-\tfrac{3}{32}\gamma^4-\tfrac{7}{64}e^2\gamma^2\}\cos(nt-\Omega)\\-e\cos(\omega-\Omega)+\{e-\tfrac{5}{4}e^3+\tfrac{1}{8}e\gamma^2\}\cos(2nt-\omega-\Omega)\\+\{\tfrac{9}{8}e^2-\tfrac{27}{16}e^4+\tfrac{3}{64}\gamma^4+\tfrac{9}{64}e^2\gamma^2\}\cos(3nt-2\omega-\Omega)\\-\{\tfrac{1}{8}\gamma^2-\tfrac{7}{128}\gamma^4-\tfrac{81}{64}e^2\gamma^2\}\cos 3(nt-\Omega)\\+\{\tfrac{1}{8}\gamma^2-\tfrac{3}{64}\gamma^4-\tfrac{29}{64}e^2\gamma^2\}\cos(nt+2\omega-3\Omega)\\-\{\tfrac{1}{8}e^2+\tfrac{1}{8}\gamma^2-\tfrac{1}{48}e^4-\tfrac{3}{64}\gamma^4-\tfrac{7}{64}e^2\gamma\}\cos(nt-2\omega+\Omega)\\+\tfrac{4}{3}e^3\cos(4nt-3\omega-\Omega)-\{\tfrac{1}{12}e^3+\tfrac{1}{8}e\gamma^2\}\cos(2nt-3\omega+\Omega)\\-\tfrac{1}{8}e\gamma^2\cos 3(\omega-\Omega)+\tfrac{1}{2}e\gamma^2\cos(2nt+\omega-3\Omega)\\-\tfrac{3}{8}e\gamma^2\cos(4nt-\omega-3\Omega)+\tfrac{625}{384}e^4\cos(5nt-4\omega-\Omega)\\-\{\tfrac{9}{128}e^4+\tfrac{9}{64}e^2\gamma^2\}\cos(3nt-4\omega+\Omega)+\tfrac{3}{128}\gamma^4\cos 5(nt-\Omega)\\+\tfrac{3}{128}\gamma^4\cos(nt+4\omega-5\Omega)-\{\tfrac{1}{128}\gamma^4+\tfrac{1}{64}e^2\gamma^2\}\cos(nt-4\omega+3\Omega)\\-\tfrac{3}{64}\gamma^4\cos(3nt+2\omega-5\Omega)-\tfrac{51}{64}e^2\gamma^2\cos(5nt-2\omega-3\Omega)\end{aligned}\right\} \tag{127}$$

If we now multiply equations (126) and (127) together, and the product by γ, we shall obtain

$$
\left.
\begin{aligned}
\frac{\gamma\cos(v-\Omega)}{r^2} = \frac{1}{a^2}&\{\gamma - \tfrac{1}{2}e^2\gamma + \tfrac{1}{8}\gamma^3 - \tfrac{3}{32}\gamma^5 - \tfrac{1}{64}e^4\gamma - \tfrac{5}{64}e^2\gamma^3\}\cos(nt-\Omega) \\
&- \{\tfrac{1}{8}\gamma^3 - \tfrac{7}{128}\gamma^5 - \tfrac{25}{64}e^2\gamma^3\}\cos 3(nt-\Omega) + \tfrac{3}{128}\gamma^5\cos 5(nt-\Omega) \\
&+ \{2e\gamma - \tfrac{3}{2}e^3\gamma + \tfrac{1}{4}e\gamma^3\}\cos(2nt-\omega-\Omega) + \tfrac{16}{3}e^3\gamma\cos(4nt-3\omega-\Omega) \\
&+ \{\tfrac{27}{8}e^2\gamma - \tfrac{27}{8}e^4\gamma + \tfrac{27}{64}e^2\gamma^3 + \tfrac{3}{64}\gamma^5\}\cos(3nt-2\omega-\Omega) \\
&+ \{\tfrac{1}{8}\gamma^3 - \tfrac{3}{64}\gamma^5 - \tfrac{11}{64}e^2\gamma^3\}\cos(nt+2\omega-3\Omega) \\
&+ \{\tfrac{1}{8}e^2\gamma - \tfrac{1}{8}\gamma^3 + \tfrac{1}{24}e^4\gamma + \tfrac{3}{64}\gamma^5 + \tfrac{5}{64}e^2\gamma^3\}\cos(nt-2\omega+\Omega) \\
&- \{\tfrac{1}{4}e\gamma^3 - \tfrac{1}{6}e^3\gamma\}\cos(2nt-3\omega+\Omega) + \tfrac{1}{2}e\gamma^3\cos(2nt+\omega-3\Omega) \\
&+ \{\tfrac{27}{128}e^4\gamma - \tfrac{27}{64}e^2\gamma^3\}\cos(3nt-4\omega+\Omega) - \tfrac{1}{2}e\gamma^3\cos(4nt-\omega-3\Omega) \\
&+ \{\tfrac{1}{64}e^2\gamma^3 - \tfrac{1}{128}\gamma^5\}\cos(nt-4\omega+3\Omega) + \tfrac{3125}{384}e^4\gamma\cos(5nt-4\omega-\Omega) \\
&+ \tfrac{3}{128}\gamma^5\cos(nt+4\omega-5\Omega) - \tfrac{3}{64}\gamma^5\cos(3nt+2\omega-5\Omega) \\
&- \tfrac{85}{64}e^2\gamma^3\cos(5nt-2\omega-3\Omega)
\end{aligned}
\right\} \quad (128)
$$

If we multiply this equation by

$$\sqrt{a\mu}\,\frac{\sqrt{1-e^2}}{\sqrt{1+\gamma^2}} = \sqrt{a\mu}\,\{1 - \tfrac{1}{2}e^2 - \tfrac{1}{2}\gamma^2 - \tfrac{1}{8}e^4 + \tfrac{3}{8}\gamma^4 + \tfrac{1}{4}e^2\gamma^2,$$

and put $\dfrac{\sqrt{\mu}}{a^{\frac{3}{2}}} = n$, we shall obtain

$$
\left.
\begin{aligned}
\frac{d\theta}{ndt} = \gamma&\{1 - e^2 - \tfrac{3}{8}\gamma^2 + \tfrac{7}{32}\gamma^4 + \tfrac{7}{64}e^4 + \tfrac{23}{64}e^2\gamma^2\}\cos(nt-\Omega) \\
&- \tfrac{1}{8}\gamma^3\{1 - \tfrac{15}{16}\gamma^2 - \tfrac{29}{8}e^2\}\cos 3(nt-\Omega) + \tfrac{3}{128}\gamma^5\cos 5(nt-\Omega) \\
&+ 2e\gamma\{1 - \tfrac{5}{4}e^2 - \tfrac{3}{8}\gamma^2\}\cos(2nt-\omega-\Omega) + \tfrac{16}{3}e^3\gamma\cos(4nt-3\omega-\Omega) \\
&+ \{\tfrac{27}{8}e^2\gamma - \tfrac{81}{16}e^4\gamma - \tfrac{81}{64}e^2\gamma^3 + \tfrac{3}{64}\gamma^5\}\cos(3nt-2\omega-\Omega) \\
&+ \{\tfrac{1}{8}\gamma^3 - \tfrac{7}{64}\gamma^5 - \tfrac{15}{64}e^2\gamma^3\}\cos(nt+2\omega-3\Omega) - \tfrac{85}{64}e^2\gamma^3\cos(5nt-2\omega-3\Omega) \\
&+ \{\tfrac{1}{8}e^2\gamma - \tfrac{1}{8}e^3 - \tfrac{1}{48}e^4\gamma + \tfrac{7}{64}\gamma^5 + \tfrac{5}{64}e^2\gamma^3\}\cos(nt-2\omega+\Omega) \\
&- \{\tfrac{1}{4}e\gamma^3 - \tfrac{1}{6}e^3\gamma\}\cos(2nt-3\omega+\Omega) + \tfrac{1}{2}e\gamma^3\cos(2nt+\omega-3\Omega) \\
&+ \{\tfrac{27}{128}e^4\gamma - \tfrac{27}{64}e^2\gamma^3\}\cos(3nt-4\omega+\Omega) - \tfrac{1}{2}e\gamma^3\cos(4nt-\omega-3\Omega) \\
&+ \{\tfrac{1}{64}e^2\gamma^3 - \tfrac{1}{128}\gamma^5\}\cos(nt-4\omega+3\Omega) + \tfrac{3125}{384}e^4\gamma\cos(5nt-4\omega-\Omega) \\
&+ \tfrac{3}{128}\gamma^5\cos(nt+4\omega-5\Omega) - \tfrac{3}{64}\gamma^5\cos(3nt+2\omega-5\Omega)
\end{aligned}
\right\} \quad (129)
$$

Equation (129) gives, by integration,

$$\left.\begin{aligned}
\theta = (\theta) &+ \gamma\{1 - e^2 - \tfrac{3}{8}\gamma^2 + \tfrac{7}{64}e^4 + \tfrac{7}{32}\gamma^4 + \tfrac{23}{64}e^2\gamma^2\}\sin(nt - \Omega) \\
&- \{\tfrac{1}{24}\gamma^3 - \tfrac{5}{128}\gamma^5 - \tfrac{33}{64}e^2\gamma^3\}\sin 3(nt - \Omega) + \tfrac{3}{640}\gamma^5 \sin 5(nt - \Omega) \\
&+ e\gamma\{1 - \tfrac{5}{4}e^2 - \tfrac{3}{8}\gamma^2\}\sin(2nt - \omega - \Omega) - \tfrac{17}{64}e^2\gamma^3\sin(5nt - 2\omega - 3\Omega) \\
&+ \{\tfrac{9}{8}e^2\gamma - \tfrac{27}{16}e^4\gamma + \tfrac{1}{64}\gamma^5 - \tfrac{27}{64}e^2\gamma^3\}\sin(3nt - 2\omega - \Omega) \\
&+ \{\tfrac{1}{8}e^2\gamma - \tfrac{1}{8}\gamma^3 - \tfrac{1}{48}e^4\gamma + \tfrac{7}{64}\gamma^5 + \tfrac{5}{64}e\gamma^3\}\sin(nt - 2\omega + \Omega) \\
&+ \{\tfrac{1}{8}\gamma^3 - \tfrac{7}{64}\gamma^5 - \tfrac{15}{64}e^2\gamma^3\}\sin(nt + 2\omega - 3\Omega) + \tfrac{4}{3}e^3\gamma\sin(4nt - 3\omega - \Omega) \\
&- \tfrac{1}{8}e\gamma^3\sin(4nt - \omega - 3\Omega) + \tfrac{1}{4}e\gamma^3\sin(2nt + \omega - 3\Omega) \\
&+ \{\tfrac{1}{12}e^3\gamma - \tfrac{1}{8}e\gamma^3\}\sin(2nt - 3\omega + \Omega) + \tfrac{625}{384}e^4\gamma\sin(5nt - 4\omega - \Omega) \\
&+ \{\tfrac{9}{128}e^4\gamma - \tfrac{9}{64}e^2\gamma^3\}\sin(3nt - 4\omega + \Omega) + \tfrac{3}{128}\gamma^5\sin(nt + 4\omega - 5\Omega) \\
&+ \{\tfrac{1}{64}e^2\gamma^3 - \tfrac{1}{128}\gamma^5\}\sin(nt - 4\omega + 3\Omega) - \tfrac{1}{64}\gamma^5\sin(3nt + 2\omega - 5\Omega)
\end{aligned}\right\}, \quad (130)$$

(θ) being the constant quantity to complete the integral. It is evident that the two expressions for the value of θ given in equations (124) and (130) will be identical if we make the constant

$$(\theta) = -e\gamma\{1 - \tfrac{1}{4}\gamma^2\}\sin(\omega - \Omega) - \tfrac{1}{8}e\gamma^3\sin 3(\omega - \Omega, \quad (131)$$

which makes equation (130) satisfy the condition that the latitude of the moon shall be equal to the latitude of the perigee of the orbit when $nt = \omega$, or when the moon is at the extremities of the transverse axis of the orbit. The perfect agreement of these two determinations of the value of θ proves conclusively that all the preceding analytical developments have been correctly made.

8. In order to show, by a few numerical examples, that the values of v, r and θ, given by equations (104), (119) and (124), are correct, and at the same time show that the development of these quantities in series to terms of the fifth order is sufficient, in the lunar theory, we shall now reduce these equations to numbers by using the values of e and γ, corresponding to the elements of the moon's orbit which were employed by DELAUNAY in reducing his equations of the lunar theory to numbers. These values are, $e = 0.05489930$, and $\gamma = 0.09004560$. And instead of giving the value of r in terms of the moon's mean distance as the unit, we have multiplied equation (119) by the constant term of the moon's parallax, supposing it to be equal to 3422″.3.

We shall therefore find

$$\left.\begin{aligned} v = nt &+ 22638.''97 \sin(nt-\omega) + 777.''074 \sin 2(nt-\omega) \\ &+ 36.''997 \sin 3(nt-\omega) + 2.''010 \sin 4(nt-\omega) + 0.''118 \sin 5(nt-\omega) \\ &- 411.''374 \sin 2(nt-\Omega) + 415.''469 \sin 2(\omega-\Omega) \\ &- 45.''255 \sin(3nt-\omega-2\Omega) + 45.''691 \sin(nt+\omega-2\Omega) \\ &- 4.''096 \sin(4nt-2\omega-2\Omega) + 0.''093 \sin(nt+3\omega-4\Omega) \\ &- 0.''186 \sin(3nt+\omega-4\Omega) + 0.''093 \sin(5nt-\omega-4\Omega) \\ &- 0.''340 \sin(5nt-3\omega-2\Omega) - 0.''848 \sin(2nt+2\omega-4\Omega) \\ &+ 0.''424 \sin 4(nt-\Omega) + 0.''424 \sin 4(\omega-\Omega) \end{aligned}\right\} . \qquad (132)$$

$$\left.\begin{aligned} \theta = {} & 18461.''23 \sin(nt-\Omega) + 1012.''716 \sin(2nt-\omega-\Omega) \\ &- 1017.''590 \sin(\omega-\Omega) + 62.''519 \sin(3nt-2\omega-\Omega) \\ &+ 18.''584 \sin(nt+2\omega-3\Omega) - 11.''662 \sin(nt-2\omega+\Omega) \\ &- 5.''993 \sin 3(nt-\Omega) + 2.''067 \sin(2nt+\omega-3\Omega) \\ &- 1.''0335 \sin(4nt-\omega-3\Omega) - 1.''0335 \sin 3(\omega-\Omega) \\ &+ 4.''098 \sin(4nt-3\omega-\Omega) + 0.''2746 \sin(5nt-4\omega-\Omega) \\ &+ 0.''0286 \sin(nt+4\omega-5\Omega) - 0.''1206 \sin(5nt-2\omega-3\Omega) \\ &- 0.''0191 \sin(3nt+2\omega-5\Omega) - 0.''0024 \sin(nt-4\omega+3\Omega) \\ &+ 0.''0057 \sin 5(nt-\Omega) - 0.''7773 \sin(2nt-3\omega+\Omega) \\ &- 0.''0520 \sin(3nt-4\omega+\Omega) \end{aligned}\right\} ; \qquad (133)$$

$$\left.\begin{aligned} \pi = {} & 3422.''300 + 187.''811 \cos(nt-\omega) + 10.''304 \cos 2(nt-\omega) \\ &+ 0.''636 \sin 3(nt-\omega) + 0.''0414 \cos 4(nt-\omega) \\ &+ 0.''0028 \cos 5(nt-\omega) \end{aligned}\right\} ; \qquad (134)$$

π denoting the moon's parallax.

We shall now compare these approximate values of v, θ and π with the values derived by an exact calculation from rigorous formulæ for the same co ordinates For this purpose we shall observe that if we denote by u the eccentric anomaly, and by $nt-\omega'$ the mean anomaly measured on the plane of the orbit, the relation between u and $nt-\omega'$ will be given by the equation

$$nt-\omega' = u - e\sin u \tag{135}$$

Then we shall have the following equation for the determination of the true anomaly $v'-\omega'$, and radius vector r,

$$\left.\begin{aligned} \tan\tfrac{1}{2}(v'-\omega') &= \frac{\sqrt{1+e}}{\sqrt{1-e}}\tan\tfrac{1}{2}u \\ r &= a(1-e\cos u) \end{aligned}\right\} \tag{136}$$

Equations (135) and (136) are entirely rigorous, and if, for a given value of the mean anomaly $nt-\omega'$, we compute the corresponding value of u with accuracy by means of equation (135), we can then compute the corresponding values of $v'-\omega'$ and r by means of equations (136) Now we also have the value of $\omega'-\Omega$ by means of the equation

$$\tan(\omega'-\Omega) = \tan(\omega-\Omega)\sec\imath, \tag{137}$$

then adding the value of $v'-\omega'$ to $\omega'-\Omega$, we shall have the true distance $v'-\Omega$, of the moon from the node measured on the orbit

Then the spherical triangle $vv'\Omega$ (see figure, Art 10, Introduction) will give

$$\left.\begin{aligned} \tan(v-\Omega) &= \tan(v'-\Omega)\cos\imath \\ \text{and}\qquad \sin\theta &= \sin(v'-\Omega)\sin\imath \end{aligned}\right\} \tag{138}$$

The formulæ (135–138) will give the rigorous values of r, v and θ corresponding to any value of the mean anomaly $nt-\omega'$

If we suppose that $\imath = 5°\ 8'\ 43''28$, and $\omega-\Omega = 30°$, we shall find that $\omega'-\Omega = 30°\ 6'\ 0''99$ Then if we designate the values of v and θ, derived from equation (138), as "observed values, and the values of the same quantities derived from equations (132) and (133) as "calculated values," we shall obtain the following table showing the values of the co ordinates v and θ for the values of the mean anomaly given in the first column of the table

Mean Anomaly.	Observed.		Calculated.		Calculated.—Observed.	
	$v - \Omega$	θ	$v - \Omega$	θ	Δv	$\Delta\theta$
30°	63° 20′ 58″.23	+4° 36′ 4″.30	63° 20′ 58″.06	+4° 36′ 4″.30	−0″.17	0″.00
40	74 18 13.61	4 57 16.05	74 18 13.72	4 57 16.06	+0.11	+0.01
60	95 45 20.32	5 7 10.39	95 45 20.51	5 7 10.34	+0.19	−0.05
80	116 27 0.21	4 36 33.14	116 27 0.22	4 36 33.10	+0.01	−0.04
100	136 19 36.09	3 33 29.25	136 19 35.97	3 33 29.18	−0.12	−0.07
130	164 46 9.63	+1 21 18.38	164 46 9.80	+1 21 18.40	+0.17	+0.02
160	192 4 24.48	−1 4 44.42	192 4 24.44	−1 4 44.40	+0.04	+0.02

The columns of residuals show that the development of the longitude and latitude in series as far as terms of the fifth order can in no case be in error to a greater extent than about 0.″2 in longitude, and 0.″1 in latitude; and it would therefore seem that the development of the lunar theory to that degree of approximation would be amply sufficient for all the purposes of astronomy, except for those terms of perturbations producing the inequalities of long period. We would here also observe that the value of the parallax given by equation (134) can in no case differ from the rigorous value of that co-ordinate to a greater extent than 0.″1; and a more perfect agreement between approximative and rigorous formulæ could hardly be expected or desired.

9. The whole of the mathematical theory of the moon's motion is contained in the preceding articles (1–8), when we neglect the consideration of the effects of the disturbing forces. The expressions for the three co-ordinates v, θ and r, given in equations (104), (124) and (119), are perfectly homogeneous in their development, every circular function or argument of the different terms of the equations being measured on the fixed plane of projection instead of the plane of the orbit. There are therefore no terms depending on the mean distance of the moon from the perigee measured on the plane of the orbit, usually called the mean anomalies; but the equivalent terms depending on the difference of the mean longitudes of the moon and of the perigee measured on the plane of projection is given instead. For this reason it is unnecessary to make any reference to the distance of the perigee from the node of the orbit measured on the plane of the orbit, but simply to designate the longitude of the perigee by the distance of its projection from the origin of longitudes and measured on the fixed plane. And therefore the expressions for these three co-ordinates which we have given are simpler than the equivalent expressions given by Delaunay, who reckons longitudes on the fixed plane, and the other arguments of his equations upon the plane of the orbit, referring them to the fixed plane in terms of certain angular functions depending on the distance between the node and perigee, and also to the mean anomalies which are measured on the plane of the orbit. Delaunay's

theory therefore requires the consideration of two planes of reference, one for the longitudes and the other for the anomalies. For this reason his formulæ are less conveniently adapted for developing the perturbations than the expressions which we have given. We may observe, however, that DELAUNAY'S expressions for v and θ may be made identical with equations (104) and (124), as far as terms of the fourth order, by changing the γ of his notation into $\frac{1}{2}\gamma(1-\frac{3}{8}\gamma^2)$, and also changing g into $(\omega-\Omega)+\frac{1}{4}\gamma^2\sin 2(\omega-\Omega)$. We may then correctly put his $l=nt-\omega$, and then the two systems of equations will be made identical. Considering the intricacy and importance of the subject, and the great extent to which it is necessary to carry the approximations, it is very essential to reduce the fundamental elements of the problem to as simple a form as possible, and it is believed that the differential equations of the moon's co ordinates, v, θ and r, given in § 1, leave little to be desired so far as the determination of these quantities is concerned. We shall, however, in order to render this chapter more complete as a basis for the further development of the lunar theory, now give the analytical expressions for the variations of the elements γ, Ω, a, e and ω, depending on the action of the disturbing forces.

10 For this purpose we shall observe that we have already found

$$\tan\Omega=\frac{c''}{c'} \tag{139}$$

If we suppose that c' and c'' are variable, from the action of the disturbing forces we shall get, by differentiating equation (139) and dividing by dt,

$$\frac{d\Omega}{dt}=\frac{\cos^2\Omega}{c'}\left\{\frac{dc''}{dt}-\frac{c''}{c'}\,\frac{dc'}{dt}\right\} \tag{140}$$

But we have $c'=c\gamma\cos\Omega$, and $c''=c\gamma\sin\Omega$, therefore equation (140) will become

$$\frac{d\Omega}{dt}=\frac{\cos\Omega}{c\gamma}\,\frac{dc''}{dt}-\frac{\sin\Omega}{c\gamma}\,\frac{dc'}{dt} \tag{141}$$

If c' and c'' are variable by reason of the disturbing forces, equations (9) and (10) will give

$$\frac{dc'}{dt}=-\left(\frac{dR}{dv}\right)\tan\theta\sin v-\left(\frac{dR}{d\theta}\right)\cos v\,, \tag{142}$$

$$\frac{dc''}{dt}=\left(\frac{dR}{dv}\right)\tan\theta\cos v-\left(\frac{dR}{d\theta}\right)\sin v \tag{143}$$

If we now substitute these values of $\frac{dc'}{dt}$ and $\frac{dc''}{dt}$ in equation (141), and also put

$$c = \frac{\sqrt{a\mu(1-e^2)}}{\sqrt{1+\gamma^2}},$$

we shall find

$$\frac{d\Omega}{dt} = \frac{\sqrt{1+\gamma^2}}{\sqrt{a\mu(1-e)}} \left\{\sin(v-\Omega)\cos(v-\Omega)\left(\frac{dR}{dv}\right) - \frac{\sin(v-\Omega)}{\gamma}\left(\frac{dR}{d\theta}\right)\right\} \tag{144}$$

In the same manner we may compute the differential variations of the elements γ, e and ω, but the process is simple, and it is unnecessary to give it in detail We shall therefore insert here the equations to be employed in the further development of the lunar theory, the solution of which cannot fail to give the moon s co ordinates with all the precision required by observation, if the inequalities of her motion are produced by the forces of gravitation

If we therefore substitute the values of c, c', c'', f and f', in equations (19), (20) and (21), we shall obtain the following differential expressions of the co ordinates v, θ and r

$$\frac{dv}{dt} = \frac{\sqrt{a\mu(1-e^2)}}{\sqrt{1+\gamma^2}\, r^2\cos^2\theta}\left\{1 - \frac{\sqrt{1+\gamma^2}}{\sqrt{a\mu(1-e^2)}}\int\left(\frac{dR}{dv}\right)dt\right\}, \tag{B}$$

$$\left.\begin{aligned}\frac{d\theta}{dt} &= \frac{\sqrt{a\mu(1-e^2)}\,\gamma}{\sqrt{1+\gamma^2}\; r^2}\cos(v-\Omega)\\ &\quad - \frac{\cos v}{r^2}\int\left\{\tan\theta\sin v\left(\frac{dR}{dv}\right) + \cos v\left(\frac{dR}{d\theta}\right)\right\}dt\\ &\quad + \frac{\sin v}{r^2}\int\left\{\tan\theta\cos v\left(\frac{dR}{dv}\right) - \sin v\left(\frac{dR}{d\theta}\right)\right\}dt\end{aligned}\right\}, \tag{C}$$

$$\frac{dr}{dt}=\frac{\sqrt{1+\gamma^2}}{\sqrt{a\mu(1-e^2)}}\mu e\cos\theta_0\cos\theta\sin(v-\omega)\left\{1-\frac{\sqrt{1+\gamma^2}}{\sqrt{a\mu(1-e^2)}}\int\left(\frac{dR}{dv}\right)dt\right\}^{-1}$$

$$+\frac{\left\{\begin{aligned}&\frac{\sqrt{1+\gamma^2}}{\sqrt{a\mu(1-e^2)}}\cos\theta\cos v\int\Big\{r^2\{d\theta\sin\theta\sin v-dv\cos\theta\cos v\}\left(\frac{dR}{dr}\right)\\&\qquad+\{2rdv\cos\theta\sin v-\frac{\cos v}{\cos\theta}dr\}\left(\frac{dR}{dv}\right)\\&\qquad+\{2rd\theta\cos\theta\sin v+dr\sin\theta\sin v\}\left(\frac{dR}{d\theta}\right)\Big\}\\&-\frac{\sqrt{1+\gamma^2}}{\sqrt{a\mu(1-e^2)}}\cos\theta\sin v\int\Big\{r^2\{d\theta\sin\theta\cos v+dv\cos\theta\sin v\}\left(\frac{dR}{dr}\right)\\&\qquad+\{2rdv\cos\theta\cos v+\frac{\sin v}{\cos\theta}dr\}\left(\frac{dR}{dv}\right)\\&\qquad+\{2rd\theta\cos\theta\cos v+dr\sin\theta\cos v\}\left(\frac{dR}{d\theta}\right)\Big\}\end{aligned}\right\}}{1-\frac{\sqrt{1+\gamma^2}}{\sqrt{a\mu(1-e^2)}}\int\left(\frac{dR}{dv}\right)dt},\tag{D}$$

$$\frac{d\Omega}{dt}=\frac{\sqrt{1+\gamma^2}}{\sqrt{a\mu(1-e^2)}}\left\{\sin(v-\Omega)\cos(v-\Omega)\left(\frac{dR}{dv}\right)-\frac{\sin(v-\Omega)}{\gamma}\left(\frac{dR}{d\theta}\right)\right\},\tag{E}$$

$$\frac{d\gamma}{dt}=\frac{\sqrt{1+\gamma^2}}{\sqrt{a\mu(1-e^2)}}\left\{\gamma\cos^2(v-\Omega)\left(\frac{dR}{dv}\right)-\cos(v-\Omega)\left(\frac{dR}{d\theta}\right)\right\},\tag{F}$$

$$\left.\begin{aligned}\mu e\cos\theta_0\frac{d\omega}{dt}&=r^2\left\{\sin\theta\sin(v-\omega)\frac{d\theta}{dt}-\cos\theta\cos(v-\omega)\frac{dv}{dt}\right\}\left(\frac{dR}{dr}\right)\\&+\left\{\frac{\cos(v-\omega)}{\cos\theta}\frac{dr}{dt}-2r\cos\theta\sin(v-\omega)\frac{dv}{dt}\right\}\left(\frac{dR}{dv}\right)\\&-\left\{\sin\theta\frac{dr}{dt}+2r\cos\theta\frac{d\theta}{dt}\right\}\sin(v-\omega)\left(\frac{dR}{d\theta}\right)\end{aligned}\right\},\tag{G}$$

$$\left.\begin{aligned}\mu\frac{de}{dt}=r^2\Big\{&\{\sin\theta_0\cos\theta-\cos\theta_0\sin\theta\cos(v-\omega)\}\frac{d\theta}{dt}-\cos\theta_0\cos\theta\sin(v-\omega)\frac{dv}{dt}\Big\}\left(\frac{dR}{dr}\right)\\ &-\left\{\frac{\cos\theta_0}{\cos\theta}\sin(v-\omega)\frac{dr}{dt}+2r\{\cos\theta_0\cos\theta\cos(v-\omega)+\sin\theta_0\sin\theta\}\frac{dv}{dt}\right\}\left(\frac{dR}{dv}\right)\\ &+\left\{\begin{aligned}&\{\sin\theta_0\cos\theta-\cos\theta_0\sin\theta\cos(v-\omega)\}\frac{dr}{dt}\\ &-2r\{\cos\theta_0\cos\theta\cos(v-\omega)+\sin\theta_0\sin\theta\}\frac{d\theta}{dt}\end{aligned}\right\}\frac{dR}{d\theta}\end{aligned}\right\}. \quad \text{(H)}$$

And lastly, equation (18) will give

$$d\frac{\mu}{a}=2\left\{\left(\frac{dR}{dr}\right)dr+\left(\frac{dR}{dv}\right)dv+\left(\frac{dR}{d\theta}\right)d\theta\right\}. \qquad \text{(I)}$$

We have already integrated equations (B), (C) and (D), when the function R is equal to nothing; and if we substitute these approximate integrals in equations (E–I), we shall obtain the approximate values of the variations of the elements of the orbit. We must then substitute the corrected elements a, e, γ, ω and ☊, in equations (B), (C) and (D), by which means we shall obtain corrected values of v, θ and r, which will contain terms depending on the first power of the disturbing force; then with the corrected values of the co-ordinates v, θ and r, together with the new elements of the orbit, we must repeat the solution of equations (E–I), by which means we shall obtain corrected values of the variations of the elements, with which we can repeat the solution of the equations which determine the co-ordinates v, θ and r. By continuing the solutions in this manner we may obtain all the terms of sensible magnitude arising from the disturbing force.

11. In bringing to a close this first chapter of a lunar theory, it will be interesting to compare the results of our analysis with the conclusions arrived at by other mathematicians, in so far as the different methods of investigation will permit. For this purpose we shall observe that both La Place and Plana in their lunar theories have first determined the mean longitude of the moon in functions of the true longitude as the independent variable, in the same manner as we have done in § 4 of this chapter; and as the co-ordinates of the fundamental differential equations employed by them have precisely the same significance as in this chapter, it would seem that the several results arrived at should be identically the same, provided the methods of development employed were equally general and the calculations had been correctly made.

La Place gives the two following equations for the determination of the mean longitude and the reciprocal of the radius vector, in functions of the

true longitude as the independent variable, which are independent of the perturbations

$$\left.\begin{aligned} nt = v &- 2e(1-\tfrac{1}{4}\gamma^2)\sin(v-\omega) + \tfrac{3}{4}e^2\sin 2(v-\omega) - \tfrac{1}{3}e^3\sin 3(v-\omega) \\ &+ \tfrac{1}{4}\gamma^2\sin 2(v-\Omega) - \tfrac{1}{4}e\gamma^2\sin(3v-\omega-2\Omega) - \tfrac{3}{4}e\gamma^2\sin(v+\omega-2\Omega) \end{aligned}\right\}, \tag{145}$$

$$\left.\begin{aligned} \frac{a}{r} = 1 + e^2 &+ e(1+e^2-\tfrac{1}{4}\gamma^2)\cos(v-\omega) + \tfrac{1}{8}e\gamma^2\cos(3v-\omega-2\Omega) \\ &+ \tfrac{1}{8}e\gamma^2\cos(v+\omega-2\Omega) \end{aligned}\right\}, \tag{146}$$

while equations (89) and (105) will give, to terms of the same order, by neglecting the constant introduced by integration,

$$\left.\begin{aligned} nt = v &- 2e\sin(v-\omega) + \tfrac{3}{4}e^2\sin 2(v-\omega) - \tfrac{1}{3}e^3\sin 3(v-\omega) \\ &+ \tfrac{1}{4}\gamma^2\sin 2(v-\Omega) - \tfrac{1}{4}e\gamma^2\sin(3v-\omega-2\Omega) - \tfrac{1}{4}\gamma^2\sin(v-3\omega+2\Omega) \end{aligned}\right\}, \tag{147}$$

$$\left.\begin{aligned} \frac{a}{r} = 1 + e^2 &+ e(1+e^2)\cos(v-\omega) + \tfrac{1}{8}e\gamma^2\cos(3v-\omega-2\Omega) \\ &- \tfrac{1}{4}e\gamma^2\cos(v+\omega-2\Omega) + \tfrac{1}{8}e\gamma\ \cos(v-3\omega+2\Omega) \end{aligned}\right\} \tag{148}$$

Comparing now the values of nt given by equations (145) and (147), we notice that the coefficients of $\sin(v-\omega)$ in the two equations differ by the quantity $\frac{1}{2}e\gamma^2$, or by a term of the third order We also see that La Place's equation contains a term of the third order depending on $\sin(v+\omega-2\Omega)$, which does not appear in our equation, while our equation contains a term of the third order depending on $\sin(v-3\omega+2\Omega)$, which does not appear in the equations of La Place or Plana except with a coefficient of the fifth order These two last mentioned differences arise directly from the introduction of the latitude of the perigee into our analysis, an element that has never been explicitly considered in the analysis of the lunar and planetary theories, but which will be found to affect all terms of the third and higher orders in the planetary theories to the same extent as in the lunar theory Indeed, the usual form of development is such that the argument of an inequality indicates the analytical order of magnitude of its coefficient, because the quantity ω is always connected with e, 2ω with e^2, etc, while Ω is always connected with γ, 2Ω with γ^2, etc, so that the combination of the terms containing the *sines* and *cosines* of angles into which these quantities enter, produce coefficients with powers and products of e and γ in which the sum of the

indices is equal to the arithmetical sum of the coefficients of ω and Ω in the expression of the angle But the introduction of the latitude θ_0, of the perigee, brings ω into the analysis unaccompanied by e, and in such manner that we have $\cos\theta_0 = \{1 + \gamma^2 \sin^2(\omega - \Omega)\}^{-\frac{1}{2}}$, which, being developed, produces the second member of equation (77), an expression in which the order of the coefficient of any *cosine* is only one half as great as the sum of the coefficients of ω and Ω in the given angle

It is therefore easy to see that the product of two or more sines or cosines having coefficients whose order of magnitude has different relations to the constant terms of the angles will completely vitiate the law in regard to the order of magnitude of any coefficient which would take place, provided the coefficients of all the sines or cosines bore the same relation to the constant terms of the angles And hence the term depending on $\sin(v + \omega - 2\Omega)$, of LA PLACE s and PLANA s analysis, disappears, and the term depending on $\sin(v - 3\omega + 2\Omega)$ comes in with a coefficient of the third order

If we now suppose that $\omega = \Omega$, in equations (145) and (147) they both reduce to the following

$$\left.\begin{aligned} v = nt - (2e + \tfrac{1}{4}e\gamma^2)\sin(v-\omega) + \tfrac{3}{4}e^2\sin 2(v-\omega) - (\tfrac{1}{3}e^3 + \tfrac{1}{4}e\gamma^2)\sin 3(v-\omega) \\ + \tfrac{1}{4}\gamma^2\sin 2(v-\Omega) \end{aligned}\right\} \quad (149)$$

In comparing equations (146) and (148), we notice that the coefficients of $\cos(v - \omega)$ differ by the term $\frac{1}{4}e\gamma^2$, while equation (146) has the term $\frac{1}{8}e\gamma^2\cos(v + \omega - 2\Omega)$, instead of the two terms, $-\frac{1}{4}e\gamma^2\cos(nt + \omega - 2\Omega) + \frac{1}{8}e\gamma^2\cos(v - 3\omega + 2\Omega)$, in equation (148) However, if we suppose that $\omega = \Omega$, they both reduce to the following

$$\frac{a}{r} = 1 + e^2 + e(1 + e^2 - \tfrac{1}{8}\gamma^2)\cos(v - \omega) + \tfrac{1}{8}e\gamma^2\cos 3(v - \omega) \quad (150)$$

If we suppose, however, that $v = \omega$, in equations (146) and (148), they will become respectively

$$\frac{a}{r} = 1 + e + e^2 + e^3 - \tfrac{1}{4}e\gamma^2 + \tfrac{1}{4}e\gamma^2\cos 2(\omega - \Omega), \quad (151)$$

and

$$\frac{a}{r} = 1 + e + e^2 + e^3 \quad (152)$$

But the true elliptical value of $\frac{a}{r}$, when the moon is in perigee, is $\frac{a}{r} = \frac{a}{a(1-e^2)} = 1 + e + e^2 + e^3$, which is the same as equation (152), and therefore

equation (151) is erroneous, unless $\omega = \Omega$, in which case it becomes the same as equation (152) If we now take the differential coefficient of equation (146), we shall obtain

$$a\frac{dr}{r^2dv} = e(1+e^2-\tfrac{1}{4}\gamma^2)\sin(v-\omega)+\tfrac{3}{8}e\gamma^2\sin(3v-\omega-2\Omega)+\tfrac{1}{8}e\gamma^2\sin(v+\omega-2\Omega) \quad (153)$$

It is evident that this value of $\frac{dr}{dv}$ never becomes equal to nothing when $v=\omega$, unless ω is also equal to Ω, or to $\Omega+90°$, a conclusion already deduced from equation (16) of the Introduction This defect, however, does not exist in equation (146), for it gives $\frac{dr}{dv}=0$, when $v=\omega$, whatever be the relation between the quantities ω and Ω